GEOMETRIC LANGLANDS IN POSITIVE CHARACTERISTIC FROM CHARACTERISTIC ZERO

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To Gérard Laumon, with gratitude for the math he brought into existence.

Abstract. We establish part of the statement of the geometric Langlands conjecture for ℓ-adic sheaves over a field of positive characteristic. Namely, we show that the category of automorphic sheaves with nilpotent singular support is equivalent to the appropriately defined category of ind-coherent sheaves on the *union of some of the connected components of the* stack of Langlands parameters.

Contents

Introduction

0.1. **What is done in this paper?**

0.1.1. This paper can be considered as a sequel to both the [AGKRRV] and [GLC] series. Namely, in [AGKRRV1, Conjecture 21.2.7] we proposed a version of the geometric Langlands conjecture (GLC) that makes sense in the context of ℓ-adic sheaves (for curves over a field of any characteristic).

Namely, it says that the (derived) category

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \tag{0.1}$$

of ℓ-adic sheaves on the moduli stack Bun_G of principal G-bundles on a (smooth and complete) curve X, *with nilpotent singular support*, is equivalent to the category

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}}),$$

where:

- $\text{LS}^{\text{restr}}_{\check{G}}$ is the prestack of $\check{G}$-local systems on X *with restricted variation*, introduced in [AGKRRV1, Sect. 1.4];
- The subscript "Nilp" stands for the restriction on the singular support (as coherent sheaves), introduced in [AG1, Sect. 11.1].

Remark 0.1.2. The fact that the subcategory (0.1) inside the ambient $\text{Shv}(\text{Bun}_G)$ is the "right object to consider" as far as automorphic sheaves are concerned was a discovery of G. Laumon in his seminal paper [Laum].

There he conjectured that all Hecke eigensheaves must belong to this subcategory. This conjecture was settled in [AGKRRV1, Theorem 14.4.3]. In fact, more is true: in *loc. cit.*, Theorem 14.4.4 it was shown that "any object of $\mathrm{Shv}(\mathrm{Bun}_G)$ that remotely looks like a Hecke eigensheaf" belongs to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

0.1.3. Unfortunately, we still cannot prove the full GLC over a field of positive characteristic. Rather, in this paper we establish a partial result. We construct a functor

$$\mathbb{L}_{\check{G}}^{\mathrm{restr}} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}), \tag{0.2}$$

and we prove.

Theorem 0.1.4. *The functor* $\mathbb{L}_{\check{G}}^{\mathrm{restr}}$ *factors via an equivalence*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\sim}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}({}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}),$$

where ${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *is the union of some of the connected components of* $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$.

This is Theorem 1.3.9(i) in the main body of the paper.

Remark 0.1.5. We remind the reader that the connected components of $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ correspond bijectively to *semi-simple* $\check{G}$-local systems (two local systems lie in the same connected component if and only if they have isomorphic semi-simplifications).

In particular, every irreducible $\check{G}$-local system lies in its own connected component (which is, however, stacky and non-reduced).

0.1.6. In addition, we prove:

Theorem 0.1.7. *If* $G = GL_n$, *then the inclusion*

$${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \subset \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$$

is an equality.

So at least for $G = GL_n$, Laumon's vision for the structure of what he called "geometric Langlands correspondence" has been fully realized.

Remark 0.1.8. One can say that for an arbitrary group G, we have not solved the most mysterious part of the Langlands conjecture: we do not know that to an irreducible $\check{G}$-local system there corresponds a non-zero Hecke eigensheaf.

Over fields of characteristic 0 we know this thanks to the Beilinson-Drinfeld construction of eigensheaves for D-modules, using localization of modules over the affine Kac-Moody algebra at the critical level[1].

0.1.9. *Assumptions on the characteristic.* The above results rely on the validity of [AGKRRV1], for which certain assumptions on the characteristic of the ground field are needed, see Sects. 14.4.1 and D.1.1 in *loc. cit.*

0.2. **Function-theoretic applications.** The equivalence established in Theorem 0.1.4 allows us to deduce information about the classical theory of automorphic functions (in the unramified case over function fields).

0.2.1. Namely, we let our ground field k be the algebraic closure of a finite field $\mathbb{F}_q$, and we assume that both X and G are defined over $\mathbb{F}_q$. In this case, the geometric objects involved in (0.2) carry an automorphism, given by the action of Frobenius.

Taking its categorical trace and applying some existing calculations (namely, ones from [AGKRRV3] and [BLR]), from Theorem 0.1.4 we obtain:

Theorem 0.2.2. *There exists an identification of vector spaces*

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \simeq \Gamma({}'\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}, \omega), \tag{0.3}$$

where:

[1]Formally speaking, this is not how the proof of GLC for D-modules given in [GLC5] proceeds; however, it does crucially rely on the localization of KM-modules, albeit slightly differently.

- $\text{Funct}_c(-, \overline{\mathbb{Q}}_\ell)$ *stands for the space of compactly supported functions;*
- $\text{LS}^{\text{arithm}}_{\check{G}} := (\text{LS}^{\text{restr}}_{\check{G}})^{\text{Frob}}$ *is the moduli space of Weil local systems on X with respect to $\check{G}$, and* $'\text{LS}^{\text{arithm}}_{\check{G}} := ('\text{LS}^{\text{restr}}_{\check{G}})^{\text{Frob}}$*;*
- ω *is the dualizing sheaf.*

This is stated as Corollary 1.5.6 in the main body of the paper.

Moreover, it follows from the construction of the isomorphism (0.3) that it is compatible with the action of the excursion algebra

$$\mathcal{A}_G := \Gamma(\text{LS}^{\text{arithm}}_{\check{G}}, \mathcal{O})$$

on both sides, and in particular, with the action of the Hecke operators.

Remark 0.2.3. Parallel to Remark 0.1.8, Theorem 0.2.2 does not settle the main mystery in classical Langlands: outside the case of $G = GL_n$, we do not yet know that to an irreducible Langlands parameter there corresponds a non-zero eigenfunction (if it existed, it would automatically be cuspidal, by the nature of the isomorphism (0.3)).

Remark 0.2.4. The stack $\text{LS}^{\text{arithm}}_{\check{G}}$ is Calabi-Yau[2], but is also highly derived in that its structure sheaf has non-trivial cohomology in infinitely many negative cohomological degrees. This implies that although there is a canonical map

$$\mathcal{O}_{\text{LS}^{\text{arithm}}_{\check{G}}} \to \omega_{\text{LS}^{\text{arithm}}_{\check{G}}}, \tag{0.4}$$

it is very far from being an isomorphism (although it is such over the *quasi-smooth* locus).

Ultimately, this phenomenon is responsible for the presence of the Arthur SL_2 in the classification of automorphic functions.

Yet, we expect that in the map

$$\Gamma(\text{LS}^{\text{arithm}}_{\check{G}}, \mathcal{O}_{\text{LS}^{\text{arithm}}_{\check{G}}}) \to \Gamma(\text{LS}^{\text{arithm}}_{\check{G}}, \omega_{\text{LS}^{\text{arithm}}_{\check{G}}}), \tag{0.5}$$

induced by (0.4), both sides are classical vector spaces (i.e., are concentrated in cohomological degree 0) and the map (0.5) itself is injective. Moreover, we expect that the intersection of the image of (0.5) with the subspace of cuspidal functions is equal to the space of *tempered* cuspidal functions (with respect to any isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$). We hope to take this up in a future work.

Remark 0.2.5. The paper [Ra2] proves an arithmetic result — the non-existence of cusp forms with certain Langlands parameters — conditional on GLC in characteristic p. Although we only obtain partial results on GLC in this paper, our results suffice for the applications in [Ra2].

0.3. **The methods.**

0.3.1. The first step is the construction of the functor (0.2). It is here that the significance of the subcategory

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$$

comes to the fore.

Namely, it turns out to be the maximal subcategory of $\text{Shv}(\text{Bun}_G)$ on which the action of the Hecke functors factors through a monoidal action of the category

$$\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}});$$

we refer to it as the "spectral action".

[2] In the sense that the determinant of the cotangent complex is the trivial line bundle.

We recover the functor (0.2) by requiring that it intertwines the actions of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ on the two sides and makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect} \\ {\scriptstyle \mathrm{coeff}^{\mathrm{Vac}}}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{IndCoh}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}^{\mathrm{restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}), \end{array}$$

where the left vertical arrow is the functor of *vacuum* Whittaker coefficient, see Sect. 9.3.1.

This follows verbatim the construction of the Langlands functor for D-modules and Betti sheaves in [GLC1, Sect. 1].

0.3.2. As a next step, we show that the functor (0.2) is an equivalence for ℓ-adic sheaves as long as the ground field over which we work has characteristic 0.

To do so, by the Lefschetz principle, we can replace the initial ground field by the field $\mathbb{C}$ of complex numbers. In the latter case, we compare the functor (0.2) for ℓ-adic sheaves with its counterpart for Betti sheaves, and we show that if the latter is an equivalence, then so is the ℓ-adic version.

Finally, we quote [GLC1], which says that the Betti version of (0.2) *is* an equivalence. This is obtained by combining the fact that the D-module version of (0.2) is an equivalence (which is the outcome of the [GLC] series) and the Riemann-Hilbert correspondence.

0.3.3. Thus, our task is to deduce Theorem 0.1.4 for a field of positive characteristic from its validity for a field of characteristic 0. We achieve this by the following procedure.

Let k be our ground field of positive characteristic (assumed algebraically closed). Let $\mathsf{R}_0 := \mathrm{Witt}(\mathsf{k})$ be the ring of Witt vectors of k, let K_0 denote the field of fractions of R_0 and let K denote the algebraic closure of K_0. Let R denote the integral closure of R_0 in K.

Given a (smooth complete) curve X_{k} over k, we *choose* its extension to a (smooth complete) curve X_{R_0} over $\mathrm{Spec}(\mathsf{R}_0)$. (Such an extension exists by a standard deformation theory argument.) Let X_{K} be the base change of X_{R_0} to K.

Note that we can identify $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$ with the union of some of the connected components of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}}$, see Sect. 3.1.3.

0.3.4. In Sect. 3 we introduce a specialization functor

$$\mathrm{Sp} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}), \tag{0.6}$$

which essentially amounts to the functor of nearby cycles.

We establish the following properties of this functor:

- It commutes with the Hecke functors;
- It commutes with the functors of Eisenstein series;
- It sends the vacuum Poincaré object[3] $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{K}}$ to the vacuum Poincaré object $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{k}}$.

The first of these properties implies that the functor Sp sends the direct summand

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$$

that has to do with $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}} \subset \mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}}$ to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$, i.e., we obtain a functor

$$\mathrm{Sp} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}). \tag{0.7}$$

[3]This is the object that corepresents the functor $\mathrm{coeff}^{\mathrm{Vac}}$.

Combined with the other properties, one shows that the functor (0.7) preserves compactness and makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) & \xrightarrow[\sim]{\mathbb{L}^{\mathrm{restr}}_{\mathsf{K}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}.\mathsf{k}}) \\ \downarrow{\scriptstyle \mathrm{Sp}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) & \xrightarrow[\mathbb{L}^{\mathrm{restr}}_{\mathsf{k}}]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}.\mathsf{k}}). \end{array}$$

0.3.5. However, this is not quite enough to deduce Theorem 0.1.4. What we need is another crucial property of the functor (0.7), which says that this functor is a *Verdier quotient*.

It turns out that there is a simple criterion for when a functor between dualizable categories is a Verdier quotient, see Lemma 5.4.5. This is a general categorical assertion, but it turns out that one can apply and check it in our situation; this is due to some rather special properties of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ established in [AGKRRV2], most notably, the categorical Künneth formula.

The fact that this criterion is satisfied for us follows from a certain geometric property of the initial functor (0.6). Namely, this property says that the natural map

$$(\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{k}}}) \to \mathrm{Sp}((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{K}}})), \tag{0.8}$$

is an isomorphism, where:

- $\Delta_{\mathcal{Y}}$ denotes the diagonal morphism of a stack $\mathcal{Y}$;
- $\underline{\mathsf{e}}_{\mathcal{Y}}$ denotes the constant sheaf on $\mathcal{Y}$.

In its turn, the fact that (0.8) is an isomorphism is equivalent to the ULA property of $(\Delta_{\mathrm{Bun}_{G,\mathsf{R}_0}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}})$ with respect to the projection $\mathrm{Bun}_{G,\mathsf{R}_0} \to \mathrm{Spec}(\mathsf{R}_0)$.

We verify the required ULA property using the Drinfeld-Lafforgue-Vinberg compactification $\overline{\mathrm{Bun}}_G$ of the diagonal map of Bun_G.

0.3.6. *Another technical result.* We now mention another result, Theorem 1.1.7, established in this paper, which is a crucial technical component for many other theorems that we prove.

Namely, Theorem 1.1.7 says that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathrm{Bun}_G)$.

This result was conjectured in [AGKRRV1][4], and it was proved in *loc. cit.* when the ground field has characteristic 0. In this paper we deduce it in the positive characteristic case using the functor (0.7).

0.4. **Structure of the paper.**

0.4.1. In Sect. 1 we construct the Langlands functor (0.2) and derive function-theoretic applications.

0.4.2. In Sect. 2 we prove that the functor (0.2) is an equivalence over a ground field of characteristic 0. Namely, we deduce this from the validity of the Betti version of GLC.

0.4.3. In Sect. 3 we stipulate the existence of the functor (0.6) with some specified properties and deduce Theorem 0.1.4.

0.4.4. In Sect. 4 we construct the functor (0.6) and establish some of its expected properties.

0.4.5. In Sect. 5 we state Theorem 5.1.3, which says that $(\Delta_{\mathrm{Bun}_{G,\mathsf{R}_0}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}})$ is ULA and show how this implies that the functor (0.7) is a Verdier quotient.

[4]This was used also as a hypothesis in [AGKRRV3] to show that the trace isomorphism $\mathrm{Tr}(\mathrm{Frob}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ reproduces the usual pointwise Frobenius map for constructible Weil sheaves.

0.4.6. In Sect. 6 we prove the local acyclicity theorems responsible for the required properties of the functor (0.6). There are four such theorems:

(i) Acyclicity of kernels defining Hecke functors. This is proved using (what essentially is) the Bott-Samelson resolution;

(ii) Acyclicity of kernels defining Eisenstein functors. This is proved using local models, called *Zastava* spaces.

(iii) Acyclicity of $(\Delta_{\mathrm{Bun}_{G,\mathsf{R}_0}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}})$. This is proved using local models for $\overline{\mathrm{Bun}}_G$, developed in [Sch].

(iv) Acyclicity of the vacuum Poincaré object $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}$.

In the present section we prove the first three of these theorems. The proofs are based on the *contraction principle*, formulated in Proposition 6.2.2.

0.4.7. In Sect. 7 we prove the acyclicity of the vacuum Poincaré object. We give two proofs. One is shorter, but it uses an additional assumption on the interaction of the characteristic of the field with $g(X)$ and G.

The second proof uses a comparison between the !-Poincaré object $\mathrm{Poinc}^{\mathrm{Vac}}_!$ with its *-counterpart $\mathrm{Poinc}^{\mathrm{Vac}}_*$, expressed by Theorem 7.5.4.

0.4.8. In Sect. 8 we prove Theorem 7.5.4, which says that the cone of the natural map

$$\mathrm{Poinc}^{\mathrm{Vac}}_! \to \mathrm{Poinc}^{\mathrm{Vac}}_*$$

belongs to the full subcategory generated by the essential images of the Eisenstein functors for proper parabolic subgroups.

Theorem 7.5.4 is of independent interest, and the proof that we give is useful as well: it consists of studying what one may call the asymptotic behavior of the Whttaker sheaf as we degenerate the character.

0.4.9. Finally, in Sect. 9 we revisit the topic of the interaction of the Langlands functor with Eisenstein series.

So far we have not mentioned Eisenstein series in the introduction, but not surprisingly, they form an integral part of the theory, and our ability to prove statements about the Langlands correspondence often relies on having good control of the Eisenstein functor.

One particular aspect of this is the computation of the Whittaker coefficient of Eisenstein series, which is performed in Theorem 9.2.7.

0.4.10. *Notations and conventions.* Notations and conventions in this paper are identical to those in [GLC1].

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1. The Langlands functor for ℓ-adic sheaves

In this section we will work over a ground field k, assumed algebraically closed, but it may have either positive characteristic or characteristic 0. Our sheaf theory $\mathrm{Shv}(-)$ (see [AGKRRV1, Sect. 1.1] for what we mean by that) will be that of (ind-)constructible $\overline{\mathbb{Q}}_\ell$-adic étale sheaves; so our field of coefficients e is $\overline{\mathbb{Q}}_\ell$.

On the geometric side, we consider the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, as defined in [AGKRRV1, Sect. 14.1].

On the spectral side we, we consider the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$, as defined in [AGKRRV1, Sect. 4.1].

The goal of this section is to construct a functor

$$\mathbb{L}^{\mathrm{restr}}_G : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

and state some results and conjectures pertaining to its properties.

1.1. **Coarse version of the functor.**

1.1.1. We start by considering the object

$$\mathrm{Poinc}^{\mathrm{Vac}}_! \in \mathrm{Shv}(\mathrm{Bun}_G)^c.$$

It is constructed by the procedure of [GLC1, Sect. 3.3] (see Sect. 7.1.1 below).

Remark 1.1.2. Note that when k has positive characteristic, $\mathrm{Poinc}^{\mathrm{Vac}}_!$ can be equivalently constructed by the procedure of [GLC1, Sect. 1.3.7], replacing the exponential D-module by the Artin-Schreier sheaf, see [GLC1, Remark 3.3.6].

1.1.3. Recall the functor

$$\mathsf{P} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

of [AGKRRV1, Sect. 15.4.5].

Denote:

$$\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp}} := \mathsf{P}(\mathrm{Poinc}^{\mathrm{Vac}}_!) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

1.1.4. Recall now that, according to [AGKRRV1, Theorem 14.3.2], the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is acted on by $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

We define the functor

$$\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

to be given by the action of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ on $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp}}$.

We will prove:

Proposition 1.1.5. *The functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$ preserves compactness.*

The proof will be given in Sect. 1.6.

1.1.6. As stated, Proposition 1.1.5 says that the functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$ sends compacts to compacts, when viewed as a functor with values in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. The following assertion was stated as [AGKRRV1, Conjecture 14.1.8]; we will prove it in this paper (see Sect. 3.1.12):

Theorem 1.1.7. *The embedding*

$$\mathrm{emb.Nilp} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$$

preserves compactness.

Thus, Proposition 1.1.5, combined with Theorem 1.1.7, say that the functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$, when viewed as taking values in $\mathrm{Shv}(\mathrm{Bun}_G)$, also preserves compactness.

1.1.8. By Proposition 1.1.5, the functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$ admits a continuous right adjoint, which we will denote by $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$.

The functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$ is $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-linear by construction. Hence, the functor $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$ acquires a structure of right-lax linearity with respect to $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

Note, however, since $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ is *semi-rigid* as a symmetric monoidal category (see [AGKRRV1, Appendix C] for what this means), we obtain that the right-lax linear structure on $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$ is actually strict.

1.1.9. We will prove:

Theorem 1.1.10. *The functor* $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$ *sends compact objects in* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *to bounded below (a.k.a. eventually coconnective) objects in* $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

The proof of this theorem will be given in Sect. 1.7.

1.2. **The Whittaker coefficient functor.**

1.2.1. Let

$$\mathrm{coeff}^{\mathrm{Vac}}:\mathrm{Shv}(\mathrm{Bun}_G)\to\mathrm{Vect}$$

be the functor co-represented by $\mathrm{Poinc}^{\mathrm{Vac}}_!$.

When $\mathrm{char}(k)$ is positive, it is given by (9.6). A variant of this holds when $\mathrm{char}(k)=0$ using the material in [GLC1, Sect. 3.3].

1.2.2. Recall the functor

$$\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-):\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})\to\mathrm{Vect},$$

see [AGKRRV1, Sect. 7.7].

Remark 1.2.3. Explicitly, the functor $\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)$ fits into the commutative diagram

$$\begin{array}{ccc}\mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) & \xrightarrow{(\Upsilon_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}})^\vee} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \\ {\scriptstyle \Gamma^{\mathrm{IndCoh}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)}\downarrow & & \downarrow{\scriptstyle \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)} \\ \mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}\,.\end{array}$$

This diagram is valid for any laft formal algebraic stack. Note, however, that for quasi-smooth[5] formal algebraic stacks (such as $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$), the top horizontal arrow is a Verdier quotient.

1.2.4. We will prove:

Proposition 1.2.5. *The composition*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\overset{\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}}{\longrightarrow}\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})\overset{\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)}{\longrightarrow}\mathrm{Vect}$$

identifies canonically with

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\overset{\mathrm{emb.Nilp}}{\hookrightarrow}\mathrm{Shv}(\mathrm{Bun}_G)\overset{\mathrm{coeff}^{\mathrm{Vac}}}{\longrightarrow}\mathrm{Vect}\,. \tag{1.1}$$

The proposition will be proved in Sect. 1.6.7.

Remark 1.2.6. Recall (see [AGKRRV1, Sect. 7.6.1]) that $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ is canonically self-dual, so that under this sel-fduality the object $\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}}\in\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ corresponds to the functor $\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)$. In particular, a $\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}}$-linear functor

$$\mathbf{C}\to\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

(for a $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-linear category $\mathbf{C}$) is uniquely recovered from the composition

$$\mathbf{C}\to\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})\overset{\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}},-)}{\longrightarrow}\mathrm{Vect}\,.$$

[5] In fact, an appropriate eventual connectivity assumption suffices.

From here here we obtain that Proposition 1.2.5 gives rise to the following characterization of the functor $\mathbb{L}^{\text{restr}}_{G,\text{coarse}}$: it is the unique functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$$

that satisfies:

- It is $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$-linear;
- Its composition with $\Gamma_!(\text{LS}^{\text{restr}}_{\check{G}}, -)$ is isomorphic to (1.1).

1.3. **Construction of the Langlands functor.**

1.3.1. Let $\mathbf{u}^{\text{spec}}$ denote the functor

$$\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}}) \overset{\Upsilon_{\text{LS}^{\text{restr}}_{\check{G}}}}{\simeq} \text{IndCoh}_{\{0\}}(\text{LS}^{\text{restr}}_{\check{G}}) \overset{-\otimes \mathfrak{l}_{\text{LS}^{\text{restr}}_{\check{G}}}}{\simeq} \text{IndCoh}_{\{0\}}(\text{LS}^{\text{restr}}_{\check{G}}) \hookrightarrow \hookrightarrow \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}}), \tag{1.2}$$

where:

- The first arrow is given by tensoring by the dualizing sheaf $\omega_{\text{LS}^{\text{restr}}_{\check{G}}}$; of $\text{LS}^{\text{restr}}_{\check{G}}$;
- $\mathfrak{l}_{\text{LS}^{\text{restr}}_{\check{G}}}$ is the graded line bundle[6] $\det(T^*(\text{LS}^{\text{restr}}_{\check{G}})^{-1})[-2(g-1)\dim(G)]$.

In what follows we will denote the composition of the first two arrows in (1.2) by $\Xi_{\text{LS}^{\text{restr}}_{\check{G}}}$. This is a functor that makes sense for any quasi-smooth formal scheme (resp., algebrac stack) $\mathcal{Z}$. If $\mathcal{Z}$ is an actual scheme (resp., algebraic stack), then $\Xi_{\mathcal{Z}}$ is the tautological functor

$$\text{QCoh}(\mathcal{Z}) \hookrightarrow \text{IndCoh}(\mathcal{Z}),$$

whose essential image is $\text{IndCoh}_{\{0\}}(\mathcal{Z})$.

Remark 1.3.2. We use the notation $\mathbf{u}^{\text{spec},R}$, rather than the more common one, namely, $\Psi_{\text{Nilp},\{0\}}$, in order to avoid the clash with the symbol for the nearby cycles functor.

Remark 1.3.3. The second arrow in (1.2) is introduced in order to make this functor compatible with the one in the de Rham and Betti versions. Note also that $\text{LS}^{\text{restr}}_{\check{G}}$ is symplectic (the symplectic structure is constructed using a choice of an invariant form on $\check{\mathfrak{g}}$) of dimension $[2(g-1)\dim(G)]$. Hence, the line bundle $\det(T^*(\text{LS}^{\text{restr}}_{\check{G}}))$ is canonically constant.

1.3.4. The functor $\mathbf{u}^{\text{spec}}$ preserves compactness and is fully faithful. Let $\mathbf{u}^{\text{spec},R}$ denote its right adjoint.

The functor $\mathbf{u}^{\text{spec}}$ is $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$-linear by construction. Hence, the functor $\mathbf{u}^{\text{spec},R}$ acquires a right-lax linear structure. By the semi-rigidity of $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$, this right-lax structure is actually strict.

1.3.5. From Theorem 1.1.10, as in [GLC1, Corollary 1.6.5], we obtain:

Corollary 1.3.6. *There exists a continuous functor*

$$\mathbb{L}^{\text{restr}}_G : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}}),$$

uniquely characterized by the following properties:

(i) *The functor* $\mathbb{L}^{\text{restr}}_G$ *sends compact objects in* $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ *to eventually coconnective objects in* $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})$*, i.e., to* $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})^{>-\infty}$*;*
(ii) $(\mathbf{u}^{\text{spec}})^R \circ \mathbb{L}^{\text{restr}}_G \simeq \mathbb{L}^{\text{restr}}_{G,\text{coarse}}$.

Furthermore, as in [GLC1, Proposition 1.7.2], we obtain:

Lemma 1.3.7. *The functor* $\mathbb{L}^{\text{restr}}_G$ *carries a unique* $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$*-linear structure, so that the induced* $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$*-linear structure on* $(\mathbf{u}^{\text{spec}})^R \circ \mathbb{L}^{\text{restr}}_G$ *is the natural* $\text{QCoh}(\text{LS}^{\text{restr}}_{\check{G}})$*-linear structure on* $\mathbb{L}^{\text{restr}}_{G,\text{coarse}}$.

[6] In fact, this line bundle is constant, i.e., is essentially a graded line over e, see Remark 1.3.3 below.

1.3.8. We are now ready to state the main result of this paper:

Main Theorem 1.3.9.
(i) *The functor* $\mathbb{L}_G^{\mathrm{restr}}$ *factors via an equivalence*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\sim}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}({}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}({}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

where ${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *is the union of some of the connected components of* $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$.
(ii) *If* $\mathrm{char}(k) = 0$, *then the inclusion* ${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \subset \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *is an equality.*
(iii) *For any* k *and* $G = GL_n$, *the inclusion* ${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \subset \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *is an equality.*

Of course, we believe that the statement of Theorem 1.3.9(i) can be strengthened:

Conjecture 1.3.10. *The inclusion* ${}'\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \subset \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *is always as equality.*

1.4. **Langlands functor and Eisenstein series.**

1.4.1. Let P^- be a standard (negative) parabolic in G and let M be its Levi quotient. Consider the Eisenstein functor

$$\mathrm{Eis}_!^- : \mathrm{Shv}(\mathrm{Bun}_M) \to \mathrm{Shv}(\mathrm{Bun}_G),$$

see [GLC3, Sect. 8.1].

Note that according to the conventions of [GLC3, Sect. 8.1.3], the definition of $\mathrm{Eis}_!^-$ includes a cohomological shift, see (9.7).

We claim:

Proposition 1.4.2. *The functor* $\mathrm{Eis}_!^-$ *sends* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$ *to* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 1.4.3. The proof given below uses a spectral description of the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. One can, however, give a purely geometric argument proving Proposition 1.4.2: Namely, one can estimate the singular support of objects of the form $\mathrm{Eis}_!^-(\mathcal{F})$ using the following assertion:

The singular support of $\jmath_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}) \in \mathrm{Shv}(\overline{\mathrm{Bun}}_{P^-})$ *is contained in the union of the conormal of the strata (for the natural stratification of* $\overline{\mathrm{Bun}}_{P^-}$*).*

This assertion can be proved using Zastava spaces in a way similar to the manipulation involved in the proof of Theorem 4.4.5 below.

Proof. Let $\mathcal{Z}$ be a prestack over e mapping to $\mathrm{LS}_{\check{M}}^{\mathrm{restr}}$. Let

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_M))$$

be the corresponding category of Hecke eigensheaves, see [AGKRRV1, Sect. 15.2].

Recall that according to [BG2] (the case of $P = B$) and its generalization to an arbitrary parabolic in [FH, Theorem 1.6.5.2], the functor

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_M)) \overset{\mathbf{oblv}_{\mathrm{Hecke}}}{\longrightarrow} \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_M) \overset{\mathrm{Id}\otimes \mathrm{Eis}_!^-}{\longrightarrow} \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \tag{1.3}$$

factors canonically as

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_M)) \overset{(\mathsf{q}^{-,\mathrm{spec}})^*\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Hecke}(\mathrm{LS}_{\check{P}^-}^{\mathrm{restr}} \underset{\mathrm{LS}_{\check{M}}^{\mathrm{restr}}}{\times} \mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_M)) \overset{\mathrm{Hecke}(\mathcal{Z},\mathrm{Eis}_!^-)}{\longrightarrow} \tag{1.4}$$

$$\to \mathrm{Hecke}(\mathrm{LS}_{\check{P}^-}^{\mathrm{restr}} \underset{\mathrm{LS}_{\check{M}}^{\mathrm{restr}}}{\times} \mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathbf{oblv}_{\mathrm{Hecke}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}^{\mathrm{restr}} \underset{\mathrm{LS}_{\check{M}}^{\mathrm{restr}}}{\times} \mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \overset{(\mathsf{q}^{-,\mathrm{spec}})_*\otimes \mathrm{Id}}{\longrightarrow}$$

$$\to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G).$$

Take $\mathcal{Z} = \mathrm{LS}_{\check{M}}^{\mathrm{restr}}$, and recall that in this case the composition

$$\mathrm{Hecke}(\mathrm{LS}_{\check{M}}^{\mathrm{restr}}, \mathrm{Shv}(\mathrm{Bun}_M)) \overset{\mathbf{oblv}_{\mathrm{Hecke}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{M}}^{\mathrm{restr}}) \otimes \mathrm{Shv}(\mathrm{Bun}_M) \overset{\Gamma_!(\mathrm{LS}_{\check{M}}^{\mathrm{restr}},-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_M) \tag{1.5}$$

is fully faithful with essential image $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$ (see [AGKRRV1, Proposition 15.5.3(a)]).

Under the identification (1.5), the original functor $\mathrm{Eis}^-_!$ identifies with the composition of (1.4) with

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \overset{\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{M}},-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G).$$

The desired assertion follows now from the fact that for any $\mathcal{Z}'$, the inclusion

$$\mathrm{Hecke}(\mathcal{Z}', \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \hookrightarrow \mathrm{Hecke}(\mathcal{Z}', \mathrm{Shv}(\mathrm{Bun}_G))$$

is an equality, see again [AGKRRV1, Proposition 15.5.3(a)].

□

1.4.4. Consider the diagram

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}} \overset{\mathsf{p}^{-,\mathrm{spec}}}{\longleftarrow} \mathrm{LS}^{\mathrm{restr}}_{\check{P}^-} \overset{\mathsf{q}^{-,\mathrm{spec}}}{\longrightarrow} \mathrm{LS}^{\mathrm{restr}}_{\check{M}},$$

where we note that the morphism $\mathsf{q}^{-,\mathrm{spec}}$ is a relative algebraic stack and is quasi-smooth.

The spectral Eisenstein functor

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \to \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

is defined to be

$$(\mathsf{p}^{-,\mathrm{spec}})^{\mathrm{IndCoh}}_* \circ (\mathsf{q}^{-,\mathrm{spec}})^{\mathrm{IndCoh},*}.$$

As in [AG1, Proposition 13.2.6], one shows that the functor $\mathrm{Eis}^{-,\mathrm{spec}}$ sends

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

1.4.5. In Sect. 9 we will prove:

Theorem 1.4.6. *The diagram*

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) & \overset{\mathbb{L}^{\mathrm{restr}}_M}{\longrightarrow} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{(N^-_P)_{\rho_P(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \underset{\mathbb{L}^{\mathrm{restr}}_G}{\longrightarrow} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \end{array} \tag{1.6}$$

commutes, where:

- *The functor* $\mathrm{Eis}^-_{!,\rho_P(\omega_X)}$ *is the precomposition of* $\mathrm{Eis}^-_!$ *with the translation functor by* $\rho_P(\omega_X) \in \mathrm{Bun}_{Z_M}$*;*
- $\delta_{(N^-_P)_{\rho_P(\omega_X)}} = \dim(\mathrm{Bun}_{(N^-_P)_{\rho_P(\omega_X)}})$*, see* [GLC3, Theorem 10.1.2].

1.5. **Consequences for the classical theory of automorphic functions.**

1.5.1. We now specialize to the case when $k = \overline{\mathbb{F}}_q$, but X and G (and hence also Bun_G) are defined over $\mathbb{F}_q$. The stack $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ carries an automorphism induced by the geometric Frobenius on X.

The following assertion was stated in [AGKRRV1] as a corollary of *loc. cit.*, Conjecture 24.6.9, and the latter was proved in [BLR, Sect. 6.4.13]:

Theorem 1.5.2. *The inclusion*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

induces an isomorphism

$$\mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})) \overset{\sim}{\to} \mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})).$$

1.5.3. A standard computation implies that

$$\mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})) \simeq \Gamma^{\mathrm{IndCoh}}((\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}}, \omega_{(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}}}).$$

Recall the notation

$$\mathrm{LS}^{\mathrm{arithm}}_{\check{G}} := (\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}}.$$

According to [AGKRRV1, Theorem 24.1.4], $\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$ is a quasi-compact algebraic stack locally almost of finite type.

Hence, Theorem 1.5.2 implies that we have a canonical isomorphism

$$\mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})) \simeq \Gamma(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}}), \tag{1.7}$$

where by a slight abuse of notation, we denote by the same symbol

$$\omega_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}})$$

the image of

$$\omega_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}} \in \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}})$$

under the functor

$$(\Upsilon_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}})^{\vee} : \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}).$$

1.5.4. Let

$$'\mathrm{LS}^{\mathrm{restr}}_{\check{G}} \subset \mathrm{LS}^{\mathrm{restr}}_{\check{G}}$$

be as in Theorem 1.3.9(i). This is a Frob-invariant substack. Set

$$'\mathrm{LS}^{\mathrm{arithm}}_{\check{G}} := ('\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}}.$$

It formally follows from Theorem 1.5.2 that we have

$$\mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}_{\mathrm{Nilp}}('\mathrm{LS}^{\mathrm{restr}}_{\check{G}})) \simeq \Gamma('\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \omega_{'\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}}). \tag{1.8}$$

1.5.5. Combining isomorphism (1.8) with Theorem 1.3.9(i) and [AGKRRV3, Theorem 0.2.6], we obtain:

Corollary 1.5.6. *There exists a canonical isomorphism*

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \simeq \Gamma('\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \omega_{'\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}}).$$

1.5.7. From Corollary 1.5.6 we obtain the following:

Corollary 1.5.8. *Let G be semi-simple, and let σ be an irreducible Weil $\check{G}$-local system. Then the space of automorphic functions on which the algebra of excursion operators acts by the character corresponding to σ is at most one-dimensional (and in the latter case is spanned by a cuspidal function).*

Proof. Recall that by [AGKRRV1, Theorem 24.1.6], an irreducible local system σ gives rise to a connected component $\mathrm{LS}^{\mathrm{arithm}}_{\check{G},\sigma}$ isomorphic to $\mathrm{pt}/\mathrm{Aut}(\sigma)$, where $\mathrm{Aut}(\sigma)$ is a finite group. In particular[7], $\omega_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G},\sigma}} \simeq \mathcal{O}_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G},\sigma}}$.

Recall now that the excursion algebra identifies with

$$H^0(\Gamma(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \mathcal{O}_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}})).$$

The map

$$\mathrm{LS}^{\mathrm{arithm}}_{\check{G},\sigma} \to \mathrm{Spec}(\Gamma(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \mathcal{O}_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G},\sigma}})),$$

induced by

$$\mathrm{LS}^{\mathrm{arithm}}_{\check{G}} \to \mathrm{Spec}(\Gamma(\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}, \mathcal{O}_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}})),$$

[7]Note also that $\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$ is canonically Calabi-Yau, i.e., the determinant line bundle of its cotangent bundle is trivialized. Hence, the restriction of $\omega_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}}$ to the quasi-smooth locus of $\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$ is canonically isomorphic to the restriction of $\mathcal{O}_{\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}}$.

identifies with

$$\mathrm{pt}\,/\,\mathrm{Aut}(\sigma) \to \mathrm{pt}\,.$$

This makes the assertion obvious.

□

Remark 1.5.9. If Conjecture 1.3.10 holds, then $'\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$ is all of $\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$.

In particular, in this case in Corollary 1.5.8, the corresponding eigenspace is exactly one-dimensional.

1.6. **Proof of Propositions 1.1.5 and 1.2.5.**

1.6.1. Write $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ as a union of its connected components

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}} = \bigsqcup_{\alpha} \mathcal{Z}_\alpha.$$

As in [AGKRRV1, Sect. 21.1], we can write each $\mathcal{Z}_\alpha$ as a (countable) colimit

$$\mathcal{Z}_\alpha \simeq \underset{n}{\mathrm{colim}}\, Z_{\alpha,n},$$

where:

- Each $Z_{\alpha,n}$ is a quasi-smooth algebraic stack;
- Each $i_{\alpha,n} : Z_{\alpha,n} \to \mathcal{Z}_\alpha$ is a *regular* closed embedding that induces an isomorphism at the reduced level.

1.6.2. The category $\mathrm{QCoh}(\mathcal{Z}_\alpha)$ is compactly generated by objects of the form

$$(i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}}) \otimes \mathcal{E},$$

where $\mathcal{E}$ is a dualizable object in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

In fact, we can take $\mathcal{E}$ to be the pullback of a vector bundle under the evaluation map

$$\mathrm{ev}_x : \mathrm{LS}^{\mathrm{restr}}_{\check{G}} \to \mathrm{pt}\,/\check{G}$$

corresponding to some chosen point $x \in X$.

1.6.3. We now begin the proof of Proposition 1.1.5.

In the above notations, it suffices to show that the functor $\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}$ sends each $(i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}})$ to a compact object of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

1.6.4. Consider the category

$$\mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

We have a pair of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-linear functors

$$((i_{\alpha,n})_* \otimes \mathrm{Id}) : \mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \rightleftarrows \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) : ((i_{\alpha,n})^! \otimes \mathrm{Id}).$$

1.6.5. Let

$$\mathsf{P}^{\mathrm{enh}}_{Z_{\alpha,n}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

be as in [AGKRRV1, Sect. 15.3.2]. I.e., this is the left adjoint to forgetful functor

$$\mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{((i_{\alpha,n})_*\otimes\mathrm{Id})}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathrm{emb.Nilp}}{\hookrightarrow} \mathrm{Shv}(\mathrm{Bun}_G),$$

see [AGKRRV1, Corollary 13.5.4].

Unwinding the construction (see [AGKRRV1, Sect. 15]), we obtain that

$$\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}}((i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}})) \simeq ((i_{\alpha,n})_* \otimes \mathrm{Id})(\mathsf{P}^{\mathrm{enh}}_{Z_{\alpha,n}}(\mathrm{Poinc}^{\mathrm{Vac}}_!)).$$

1.6.6. Now the object $\mathsf{P}^{\mathrm{enh}}_{Z_{\alpha,n}}(\mathrm{Poinc}^{\mathrm{Vac}}_!)$ is compact in $\mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (being the value of a left adjoint on a compact object).

Finally, the functor $(i_{\alpha,n})_* \otimes \mathrm{Id}$ preserves compactness since it admits a continuous right adjoint. □[Proposition 1.1.5]

1.6.7. *Proof of Proposition 1.2.5.* Note that

$$\Gamma_!(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}, -) \simeq \underset{\alpha}{\oplus} \Gamma_!(\mathcal{Z}_\alpha, (-)|_{\mathcal{Z}_\alpha}),$$

while the functor

$$\Gamma_!(\mathcal{Z}_\alpha, -) : \mathrm{QCoh}(Z_\alpha) \to \mathrm{Vect}$$

can be written as

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}((i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}}), -).$$

Thus, we can rewrite the composition

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \overset{\Gamma_!(\mathcal{Z}_\alpha,-)}{\longrightarrow} \mathrm{Vect}$$

as

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}((i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}}), \mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}(-)),$$

and hence by adjunction as

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}(\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp}} \circ (i_{\alpha,n})_*(\mathcal{O}_{Z_{\alpha,n}}), -).$$

By Sect. 1.6.5, we can rewrite the latter expression as

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}\left(((i_{\alpha,n})_* \otimes \mathrm{Id})(\mathsf{P}^{\mathrm{enh}}_{Z_{\alpha,n}}(\mathrm{Poinc}^{\mathrm{Vac}}_!)), -\right),$$

and further as

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{QCoh}(Z_{\alpha,n}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}\left(\mathsf{P}^{\mathrm{enh}}_{Z_{\alpha,n}}(\mathrm{Poinc}^{\mathrm{Vac}}_!), ((i_{\alpha,n})^! \otimes \mathrm{Id})(-)\right),$$

and again by adjunction

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}\left(\mathrm{Poinc}^{\mathrm{Vac}}_!, \mathrm{emb.\,Nilp} \circ ((i_{\alpha,n})_* \otimes \mathrm{Id}) \circ ((i_{\alpha,n})^! \otimes \mathrm{Id})(-)\right).$$

Since $\mathrm{Poinc}^{\mathrm{Vac}}_!$ is compact, the latter expression identifies with

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}\left(\mathrm{Poinc}^{\mathrm{Vac}}_!, \mathrm{emb.\,Nilp}\left(\underset{n}{\mathrm{colim}}\, ((i_{\alpha,n})_* \otimes \mathrm{Id}) \circ ((i_{\alpha,n})^! \otimes \mathrm{Id})(-)\right)\right).$$

The required assertion follows now from the fact that the natural transformation

$$\underset{\alpha}{\oplus}\, \underset{n}{\mathrm{colim}}\, ((i_{\alpha,n})_* \otimes \mathrm{Id}) \circ ((i_{\alpha,n})^! \otimes \mathrm{Id}) \to \mathrm{Id}$$

on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is an isomorphism.

□[Proposition 1.2.5]

1.7. **Proof of Theorem 1.1.10.** The proof will largely follow [GLC1, Sect. 2].

1.7.1. For the proof we will assume the validity of Theorem 1.1.7, which will be proved independently.

Assuming this theorem and using [GLC2, Sect. 2.2], we obtain that if $\mathcal{M} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compact, then

$$\mathrm{emb.\,Nilp}(\mathcal{M}) \in \mathrm{Shv}(\mathrm{Bun}_G)$$

is bounded below.

Hence, it is enough to show that the functor $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$ has a cohomological amplitude bounded on the left, i.e., there exists an integer d such that $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}[-d]$ is left t-exact.

1.7.2. Choose a point $x \in X$, and set

$$\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x} := \mathrm{LS}_{\check{G}}^{\mathrm{restr}} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt},$$

where $\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \to \mathrm{pt}/\check{G}$ is the evaluation map ev_x.

According to [AGKRRV1, Theorem 1.6.3], the prestack $\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x}$ is a disjoint union of formal affine schemes. In particular, the functor

$$\Gamma_!(\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x}, -) : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x}) \to \mathrm{Vect}$$

is t-exact and conservative.

1.7.3. Note also that we have a canonical isomorphism

$$\Gamma_!(\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x}, \pi^*(-)) \simeq \Gamma_!(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}, \mathrm{ev}_x^*(R_{\check{G}}) \otimes (-)),$$

where:

- πv denotes the projection $\mathrm{LS}_{\check{G}}^{\mathrm{restr,rigid}_x} \to \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$;
- $R_{\check{G}} \in \mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{G})$ is the regular representation.

Hence, we obtain that it is enough to show that the functors

$$\Gamma_! \left(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}, \mathrm{ev}_x^*(V) \otimes \mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}}(-)\right), \quad V \in \mathrm{Rep}(\check{G})^{\heartsuit}.$$

have cohomological amplitudes uniformly bounded on the left.

1.7.4. By Proposition 1.2.5, we rewrite the above functor as

$$\mathrm{coeff}^{\mathrm{Vac}} \circ \mathrm{H}_{V,x}(-), \tag{1.9}$$

where $\mathrm{H}_{V,x}$ is the Hecke endofunctor of $\mathrm{Shv}(\mathrm{Bun}_G)$ corresponding to the chosen x and V.

We will show that the functors (1.9) (for $V \in \mathrm{Rep}(\check{G})$) have cohomological amplitudes uniformly bounded on the left on all of $\mathrm{Shv}(\mathrm{Bun}_G)$ (and not just $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$).

1.7.5. Consider the *tempered subcategory*

$$\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}} \overset{\mathbf{u}}{\hookrightarrow} \mathrm{Shv}(\mathrm{Bun}_G)$$

as defined in [FR, Sect. 7] (using the choice of $x \in X$). The embedding $\mathbf{u}$ admits a continuous right adjoint, denoted $\mathbf{u}^R$.

The category $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}}$ carries a uniquely defined t-structure for which the functor $\mathbf{u}^R$ is t-exact, see [FR, Sect. 7.2].

Another key feature of this t-structure is that the Hecke functors $\mathrm{H}_{V,x}$ descend to $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}}$ and are t-exact on it (for $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$), see [FR, Theorem 7.1.0.1].

1.7.6. Note that since $\mathrm{Poinc}_!^{\mathrm{Vac}} \in \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}}$, the functor $\mathrm{coeff}^{\mathrm{Vac}}$ factors canonically as

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathbf{u}^R}{\to} \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}} \overset{\mathrm{coeff}^{\mathrm{Vac}}_{\mathrm{temp}}}{\longrightarrow} \mathrm{Vect}\,.$$

Hence, it is enough to show that the functors

$$\mathrm{coeff}^{\mathrm{Vac}}_{\mathrm{temp}} \circ \mathrm{H}_{V,x}(-) : \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{Vect}$$

have cohomological amplitudes uniformly bounded on the left.

1.7.7. By the t-exactness of the Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}}$, it suffices to show that the functor

$$\mathrm{coeff}^{\mathrm{Vac}}_{\mathrm{temp}} : \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{Vect}$$

has a cohomological amplitude bounded on the left.

This is in turn equivalent to the functor

$$\mathrm{coeff}^{\mathrm{Vac}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

having a cohomological amplitude bounded on the left, which is obvious.

□[Theorem 1.1.10]

2. Proof of Theorem 1.3.9 in characteristic 0

In this section we will show that when $\mathrm{char}(k) = 0$, the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbb{L}^{\mathrm{restr}}_G}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

is an equivalence.

2.1. **Constructible Betti geometric Langlands.** In this subsection we take k to be the field of complex numbers $\mathbb{C}$, and we will work with the sheaf theory denoted $\mathrm{Shv}^{\mathrm{Betti,constr}}(-)$ of (ind-)constructible Betti sheaves with coefficients in an arbitrary field e of characteristic 0.

2.1.1. Let $\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}$ be the moduli space of local systems with restricted variation, defined using the constructible sheaf theory $\mathrm{Shv}^{\mathrm{Betti,constr}}(-)$.

The material in Sects. 1.1-1.4 applies verbatim and we obtain a functor

$$\mathbb{L}^{\mathrm{Betti,restr}}_G : \mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}).$$

2.1.2. We claim:

Theorem 2.1.3. *The functor $\mathbb{L}^{\mathrm{Betti,restr}}_G$ is an equivalence.*

Proof. The above functor $\mathbb{L}^{\mathrm{Betti,restr}}_G$ is the same as the functor denoted by the same symbol in [GLC1, Sect. 3.5.3].

Now the assertion follows from the validity of the full de Rham version of the geometric Langlands conjecture (proved in [GLC5]) combined with [GLC1, Theorem 3.5.6].

□

2.2. **Betti vs étale comparison.** In this subsection we continue to assume that $k = \mathbb{C}$, and we will take $\mathsf{e} := \overline{\mathbb{Q}}_\ell$. We will work with two sheaf theories:

One is $\mathrm{Shv}^{\mathrm{et}}(-) =: \mathrm{Shv}(-)$ of (ind-)constructible $\overline{\mathbb{Q}}_\ell$-adic étale sheaves, considered in Sect. 1.

The other is $\mathrm{Shv}^{\mathrm{Betti,constr}}(-)$ considered in Sect. 2.1 above.

We will show that the validity of Theorem 2.1.3 implies the validity of Theorem 1.3.9(ii) (for $k = \mathbb{C}$).

2.2.1. Note that we have a fully faithful natural transformation between the sheaf theories

$$(\mathrm{et} \to \mathrm{Betti}) : \mathrm{Shv}^{\mathrm{et}}(-) \to \mathrm{Shv}^{\mathrm{Betti,constr}}(-) \tag{2.1}$$

that commutes with both !- and *- direct and inverse images, see [SGA4(3), Theorems XI.4.4, XVI.4.1 and XVII.5.3.3].

At the level of compact objets (i.e., constructible sheaves) on affine schemes, its essential image is characterized as follows:

Let $\mathcal{F}$ be an object of $\mathrm{Shv}^{\mathrm{Betti,constr}}(Y)^c$, and let Y_α be a decomposition of Y into smooth locally closed subsets, such that the restrictions (either *- or !-) of $\mathcal{F}$ to the strata are lisse. Consider the individual cohomology sheaves $H^i(\mathcal{F}|_{Y_\alpha})$ as representations $V_{\alpha,i}$ of $\pi_1(Y_\alpha)$ (for some chosen base point).

Then $\mathcal{F}$ lies in the essential image of $(\mathrm{et} \to \mathrm{Betti}))_Y$ if and only if each $V_{\alpha,i}$ admits a $\pi_1(Y_\alpha)$-invariant lattice with respect to $\mathcal{O}_E \subset E \subset \overline{\mathbb{Q}}_\ell$ for a finite extension $E \supseteq \mathbb{Q}_\ell$.

2.2.2. In particular, the natural transformation (2.1) induces a fully faithful functor

$$(\mathrm{et}\to\mathrm{Betti})_{\mathrm{Bun}_G}:\mathrm{Shv}^{\mathrm{et}}(\mathrm{Bun}_G)\to\mathrm{Shv}^{\mathrm{Betti,constr}}(\mathrm{Bun}_G),$$

which restricts to a (fully faithful) functor

$$\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

2.2.3. Recall that Theorem 1.1.7 has been proved for

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow\mathrm{Shv}^{\mathrm{Betti,constr}}(\mathrm{Bun}_G)$$

in [AGKRRV1, Theorem 16.4.10]. Hence, it holds for

$$\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow\mathrm{Shv}^{\mathrm{et}}(\mathrm{Bun}_G)$$

as well.

2.2.4. Let

$$\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}\ \text{ and }\ \mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}$$

be the two versions of the be the moduli space of local systems with restricted variation.

The functor

$$(\mathrm{et}\to\mathrm{Betti})_X:\mathrm{QLisse}^{\mathrm{et}}(X)\to\mathrm{QLisse}(X)^{\mathrm{Betti,constr}}$$

is symmetric monoidal, and hence induces a map between the corresponding moduli spaces

$$(\mathrm{et}\to\mathrm{Betti})_{\mathrm{LS}_{\check{G}}}:\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}\to\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}. \tag{2.2}$$

We claim:

Lemma 2.2.5. *The map (2.2) factors through an isomorphism from $\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}$ to the disjoint union of some of the connected components of $\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}$.*

Proof. Follows from the description of essential image of the natural transformation (2.1) in Sect. 2.2.1, see [AGKRRV1, Sect. 9.5.8].

□

Remark 2.2.6. An assertion parallel to Lemma 2.2.5 holds for any full symmetric monoidal subcategory of $\mathrm{Lisse}(X)^{\heartsuit}$.

2.2.7. Thus, we can regard

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})}{\otimes}\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

as a full subcategory (in fact, a direct summand) of $\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and it is easy to see that the functor

$$(\mathrm{et}\to\mathrm{Betti})_{\mathrm{Bun}_G}:\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

lands in it: this follows from the fact that this functor is compatible with the actions of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})$, viewed as a direct factor of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}$ on the two sides.

Similarly, direct image along $(\mathrm{et}\to\mathrm{Betti})_{\mathrm{LS}_{\check{G}}}$ is an equivalence

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})\overset{\sim}{\to}\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})}{\otimes}\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

Furthermore, it follows from Proposition 1.2.5 that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{(\mathrm{et}\to\mathrm{Betti})_{\mathrm{Bun}_G}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})}{\otimes} \mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\ \Big\downarrow{\scriptstyle \mathbb{L}^{\mathrm{et,restr}}_G} & & \sim\Big\downarrow{\scriptstyle \mathrm{Id}\otimes\mathbb{L}^{\mathrm{Betti,restr}}_G} \\ \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) & \underset{\sim}{\xrightarrow{(\mathrm{et}\to\mathrm{Betti})_{\mathrm{LS}_{\check{G}}}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}), \end{array}$$

in which the top horizontal arrow is fully faithful.

Hence, we obtain that the functor $\mathbb{L}^{\mathrm{et,restr}}_G$ is fully faithful. Thus, in order to prove that it is an equivalence, it remains to show that the essential image of $\mathbb{L}^{\mathrm{et,restr}}_G$ generates the target.

2.2.8. Write

$$\mathbb{L}^{\mathrm{et,restr}}_G = (\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{red}} \sqcup (\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{irred}};$$

these are unions of connected components that correspond to reducible (resp., irreducible) local systems.

It suffices to show that the essential image of each of the corresponding functors

$$\mathrm{QCoh}((\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{red}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})}{\otimes} \mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathrm{Id}\otimes\mathbb{L}^{\mathrm{et,restr}}_G}{\longrightarrow}$$
$$\to \mathrm{QCoh}((\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{red}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{red}})$$

and

$$\mathrm{QCoh}((\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{irred}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})}{\otimes} \mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathrm{Id}\otimes\mathbb{L}^{\mathrm{et,restr}}_G}{\longrightarrow}$$
$$\to \mathrm{QCoh}((\mathbb{L}^{\mathrm{et,restr}}_G)_{\mathrm{irred}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{irrred}})$$

generates its target category.

This is in turn equivalent to showing that the essential image of the each of the functors

$$\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbb{L}^{\mathrm{et,restr}}_G}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \overset{\mathrm{restriction}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{red}}) \tag{2.3}$$

and

$$\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbb{L}^{\mathrm{et,restr}}_G}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \overset{\mathrm{restriction}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{irred}}) \tag{2.4}$$

generates the target category.

2.2.9. We first prove the assertion for (2.3). By induction on the semi-simple rank, we may assume that Theorem 1.3.9(ii) holds for proper Levi subgroups of G.

Note that[8] by [AG1, Theorem 13.3.6], the union of the essential images of the functors

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})$$

for proper Levi subgroups generates $\mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{red}})$, viewed as a full subcategory in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})$.

Hence, the required generation assertion follows from Theorem 1.4.6.

[8]In *loc. cit.* this is proved in the de Rham context, but the argument applies in any sheaf-theoretic context.

2.2.10. We now show that (2.4) generates the target category. Note that the functor $\mathbf{u}^{\mathrm{spec}}$ (see Sect. 1.3.1) induces an *equivalence*

$$\mathrm{QCoh}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{irred}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{irred}}).$$

Hence, it is enough to show that the essential image of the functor

$$\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbb{L}^{\mathrm{et,restr}}_{G,\mathrm{coarse}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \overset{\mathrm{restriction}}{\longrightarrow} \mathrm{QCoh}((\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})_{\mathrm{irred}})$$

generates the target category.

We will show that the essential image of the functor $\mathbb{L}^{\mathrm{et,restr}}_{G,\mathrm{coarse}}$ itself generates the target category. First, we note that this functor is fully faithful (indeed, our situation embeds fully faithfully into the Betti situation, where the functor in question is an equivalence). Hence, it is enough to show that its *left*[9] adjoint, i.e., the functor $\mathbb{L}^{\mathrm{et,restr},L}_{G,\mathrm{temp}}$, is conservative.

Note, however, that the diagram

$$\begin{array}{ccc}
\mathrm{Shv}^{\mathrm{et}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}^{\mathrm{et,restr},L}_{G,\mathrm{temp}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}) \\
{\scriptstyle (\mathrm{et}\to\mathrm{Betti})_{\mathrm{Bun}_G}}\downarrow & & \downarrow{\scriptstyle (\mathrm{et}\to\mathrm{Betti})_{\mathrm{LS}_{\check{G}}}} \\
\mathrm{Shv}^{\mathrm{Betti,constr}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \underset{\sim}{\xleftarrow{\mathbb{L}^{\mathrm{Betti,restr},L}_{G,\mathrm{temp}}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})
\end{array}$$

is commutative: indeed, the two circuits are compatible with the spectral action $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}})$ (which we think as a direct factor of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})$) and send $\mathcal{O}_{\mathrm{LS}^{\mathrm{et,restr}}_{\check{G}}}$ to (the Betti version of) $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp}}$.

Since the right vertical arrow in the diagram is conservative, and the bottom horizontal arrow is an equivalence, we obtain that the top horizontal arrow is conservative, as required.

□[Theorem 1.3.9(ii) for $k = \mathbb{C}$]

2.3. **Proof of Theorem 1.3.9(ii) for an arbitrary k of char. 0.** The proof will use the Lefschetz principle.

2.3.1. Note that if $k_1 \subset k_2$ is an extension of algebraically closed fields of characteristic 0, for a prestack $\mathcal{Z}_1$ over k_1 and its base change $\mathcal{Z}_2$ to k_2, the pullback functor

$$\mathrm{Lisse}(\mathcal{Z}_1) \to \mathrm{Lisse}(\mathcal{Z}_2) \tag{2.5}$$

is an equivalence, see [SGA1, Proposition XIII.4.3].

This formally implies that the pullback functor

$$\mathrm{Shv}(\mathcal{Z}_1) \to \mathrm{Shv}(\mathcal{Z}_2)$$

is fully faithful.

Let $\mathcal{N}_1 \subset T^*(\mathcal{Z}_1)$ be a conical Lagrangian subset. It follows from (2.5) that the pullback functor

$$\mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Z}_1) \to \mathrm{Shv}_{\mathcal{N}_2}(\mathcal{Z}_2)$$

is an equivalence.

2.3.2. In particular, for a curve X_1 defined over k_1, and its base change X_2 to k_2, the map

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(X_1) \to \mathrm{LS}^{\mathrm{restr}}_{\check{G}}(X_2)$$

is an isomorphism (as prestacks over $\overline{\mathbb{Q}}_\ell$).

---

[9]Note that without fully faithfulness, generation of the target is equivalent to the fact that the *right* adjoint be consevative.

2.3.3. Given the curve X over k, let k' be a countably generated field over which it is defined; denote the resulting curve over k' by X'.

We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G(X')) & \xrightarrow{\mathbb{L}^{\mathrm{restr}}_G(X')} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(X')) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G(X)) & \xrightarrow{\mathbb{L}^{\mathrm{restr}}_G(X)} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(X)). \end{array}$$

Hence, the validity of Theorem 1.3.9(ii) over k is equivalent to its validity over k'.

2.3.4. Embedding k' into $\mathbb{C}$, we obtain that the validity of Theorem 1.3.9(ii) over k' is equivalent to its validity over $\mathbb{C}$. However, the latter has been established in Sect. 2.2 above.

$\square$[Theorem 1.3.9(ii)]

3. The specialization functor

In this section we introduce a procedure that will allow us to deduce information about the Langlands functor in characteristic p from its counterpart in characteristic 0.

This procedure essentially amounts to taking nearby cycles for a family of curves over a DVR.

3.1. **Axiomatics for the functor.**

3.1.1. Let k be an (algebraically closed) field of positive characteristic. Let $\mathsf{R}_0 := \mathrm{Witt}(\mathsf{k})$ be the ring of Witt vectors of k, let K_0 denote the field of fractions of R_0 and let K denote the algebraic closure of K_0. Let R denote the integral closure of R_0 in K.

Given a (smooth complete) curve X_{k} over k, we can *choose* its extension to a (smooth complete) curve X_{R_0} over $\mathrm{Spec}(\mathsf{R}_0)$. Let X_{K} be the base change of X_{R_0} to K.

3.1.2. *Notational convention.* We will insert subscripts k, R_0 or K into the corresponding geometric objects in order to specify which situation we are working with.

A symbol without such a subscript (e.g., just Bun_G) means that the discussion applies to any of the above situations.

3.1.3. Note that restriction along $X_{\mathsf{k}} \to X_{\mathsf{R}_0}$ defines an equivalence

$$\mathrm{QLisse}(X_{\mathsf{R}_0}) \simeq \mathrm{QLisse}(X_{\mathsf{k}}).$$

Restriction along $X_{\mathsf{K}} \to X_{\mathsf{R}_0}$ defines a *fully faithful* functor

$$\mathrm{QLisse}(X_{\mathsf{R}}) \to \mathrm{QLisse}(X_{\mathsf{K}}).$$

From here we obtain an embedding

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}} \overset{\iota}{\hookrightarrow} \mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}},$$

which identifies $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$ with the union of *some of* the connected components of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}}$.

3.1.4. Consider the corresponding moduli stacks

$$\mathrm{Bun}_{G,\mathsf{k}} \text{ and } \mathrm{Bun}_{G,\mathsf{K}}.$$

Consider the full subcategory (in fact, a direct summand)[10]

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) := \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}}) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}}).$$

[10]We emphasize that there is no such object as $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$; only $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$.

3.1.5. We will prove:

Theorem 3.1.6. *There exists a functor*

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \tag{3.1}$$

with the following properties:

- (A) *The functor* $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ *is a Verdier quotient.*
- (B) *The functor* $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ *intertwines the actions of* $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})$.
- (C) *The functor* $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ *sends* $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}}$ *to* $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{k}}$, *where* $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}}$ *is the direct summand of* $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K}}$ *that belongs to* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$.
- (D) *The functor* $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ *makes the diagrams*

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{K},\mathsf{k}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k},M}} & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{k}}) \\ {\scriptstyle \mathrm{Eis}^-_{!,\mathsf{K}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^-_{!,\mathsf{k}}} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k},G}} & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \end{array}$$

  commute.
- (E) *The functor* $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ *is t-exact.*

This theorem will be proved in the course of Sects. 5-8. For the rest of this section we will assume Theorem 3.1.6. We will derive some further properties of the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ of (3.1), as well as consequences for the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$ that follow from the above properties.

3.1.7. We claim that the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ of (3.1) preserves compactness.

Indeed, the validity of Theorem 1.3.9 for K implies that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$ has compact generators of the form:

(i) $\mathcal{E}\star\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}}$, where $\mathcal{E}$ is a dualizable object in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})$;

(ii) $\mathrm{Eis}^-_!(\mathcal{F}_M)$ for $\mathcal{F}_M\in\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{K},\mathsf{k}})^c$.

Now, objects of the form

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{E}\star\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}})$$

are compact in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$ by Properties (B) and (C), and objects of the form

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathrm{Eis}^-_!(\mathcal{F}_M))$$

are compact in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$ by Property (D) and induction on semi-simple rank.

3.1.8. Note that combining with Property (A), we obtain:

Corollary 3.1.9. *Objects of the form*

(i) $\mathcal{E}\star\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{k}}$, $\mathcal{E}$ *is a dualizable object in* $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})$*;*

(ii) $\mathrm{Eis}^-_!(\mathcal{F}_M)$, $\mathcal{F}_M\in\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{k}})^c$

compactly generate $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$.

Remark 3.1.10. Note that Corollary 3.1.9 is as an extension to positive characteristic of the main result of [FR].

One can similarly show that objects of type (i) in Corollary 3.1.9 generate the *tempered* subcategory of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{k}})$.

Remark 3.1.11. We conjecture that the entire $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$ is generated by objects of the form:

(i) $\mathcal{F} \star \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{k}}$, $\mathcal{F} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$,

(ii) $\mathrm{Eis}^-_!(\mathcal{F}_M)$, $\mathcal{F}_M \in \mathrm{Shv}(\mathrm{Bun}_{M,\mathsf{k}})^c$.

We can prove this over a ground field of characteristic 0, but so far not over a field of positive characteristic.

3.1.12. *Proof of Theorem 1.1.7.* We can now deduce Theorem 1.1.7 for k.

We need to show that compact generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$ are compact as objects of $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$.

By Corollary 3.1.9, it suffices to show to show that the objects

$$\mathcal{E} \star \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{k}} \simeq \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{E} \star \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}})$$

and

$$\mathrm{Eis}^-_!(\mathcal{F}_M), \quad \mathcal{F}_M \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{k}})^c$$

are compact in $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$.

For objects of the second type, this is clear by induction on the semi-simple rank, since the functor $\mathrm{Eis}^-_!$ preserves compactness. Thus, it remains to deal with objects of the first type.

According to [AGKRRV1, Proposition 16.4.7], the validity of Theorem 1.1.7 (in a given context) is equivalent to the fact that the compact generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ are eventually coconnective.

Hence, this property holds for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$ by Sect. 2.2.3. In particular, the objects $\mathcal{E} \star \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K},\mathsf{k}}$ are eventually coconnective. Now Property (E) implies that the objects

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{E} \star \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp},\mathsf{K}})$$

are also eventually coconnective.

□[Theorem 1.1.7]

3.2. **Proof of Theorem 1.3.9(i).**

In this subsection we will show that the existence of the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ with Properties (A)-(E) specified in Theorem 3.1.6 allows us to deduce Theorem 1.3.9(i) from Theorem 1.3.9(ii).

3.2.1. First, note that the validity of Theorem 1.3.9(ii) for K implies that the functor $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{K}}$ induces an equivalence

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \overset{\sim}{\to} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}});$$

denote it by $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{K},\mathsf{k}}$.

3.2.2. Set

$$'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}} := \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{K},\mathsf{k}})^{-1}, \quad \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}). \tag{3.2}$$

Note that it follows from Property (C) that

$$'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}} \circ \mathbf{u}^{\mathrm{spec}} \simeq \mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp},k}. \tag{3.3}$$

3.2.3. We will prove:

Proposition 3.2.4. *The functor* $'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}$ *is the left adjoint of* $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$.

The proof will be given in Sect. 3.3 below. We proceed with the proof of Theorem 1.3.9(i). Assuming the proposition, we will denote

$$\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}} := {}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}.$$

3.2.5. As a formal corollary of Proposition 3.2.4 combined with Property (A), we obtain:

Corollary 3.2.6. *The functor* $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ *is fully faithful.*

Thus, to prove Theorem 1.3.9(ii) it remains to show that the essential image of $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ equals

$$\mathrm{IndCoh}_{\mathrm{Nilp}}({}'\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}),$$

where ${}'\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$ is the disjoint union of some of the connected components of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$.

3.2.7. Since the functors involved are $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})$-linear, we can work with one connected component of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$ at a time. I.e., we need to show that for a given connected component $\mathcal{Z}_\alpha$ of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$, either the adjoint functors

$$\mathbb{L}^{\mathrm{restr}}_{\check{G},\mathsf{k},\alpha}:\mathrm{QCoh}(\mathcal{Z}_\alpha)\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes}\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})\leftrightarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathcal{Z}_\alpha):\mathbb{L}^{\mathrm{restr},L}_{G,k,\alpha} \tag{3.4}$$

are mutually inverse equivalences, or the left-hand side is zero.

We consider separately the cases when a given connected component corresponds to reducible or irreducible local systems.

3.2.8. We start with the irreducible case. Note that the inclusion

$$\mathbf{u}^{\mathrm{spec}}:\mathrm{QCoh}(\mathcal{Z}_\alpha)\to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathcal{Z}_\alpha)$$

is an equality.

Hence, the composition

$$\mathbb{L}^{\mathrm{restr}}_{\check{G},\mathsf{k},\alpha}\circ \mathbb{L}^{\mathrm{restr},L}_{G,k,\alpha},$$

being a $\mathrm{QCoh}(\mathcal{Z}_\alpha)$-linear endofunctor of $\mathrm{QCoh}(\mathcal{Z}_\alpha)$, is given by tensoring by a unital algebra object[11]

$$\mathcal{A}_\alpha\in \mathrm{QCoh}(\mathcal{Z}_\alpha).$$

We need to show that either the unit map

$$\mathcal{O}_{\mathcal{Z}_\alpha}\to \mathcal{A}_\alpha \tag{3.5}$$

is an isomorphism, or $\mathcal{A}_\alpha=0$.

By Barr-Beck-Lurie, the right adjoint in (3.4) identifies the left-hand side with

$$\mathcal{A}_\alpha\text{-mod}(\mathrm{QCoh}(\mathcal{Z}_\alpha)).$$

By Corollary 3.2.6, the forgetful functor

$$\mathcal{A}_\alpha\text{-mod}(\mathrm{QCoh}(\mathcal{Z}_\alpha))\to \mathrm{QCoh}(\mathcal{Z}_\alpha) \tag{3.6}$$

is fully faithful.

Let

$$i_\sigma:\mathrm{pt}\to\mathcal{Z}_\alpha$$

correspond to the (unique) closed point of $\mathcal{Z}_\alpha$.

Applying to (3.6) the operation $\mathrm{Vect}\underset{i_\sigma^*,\mathrm{QCoh}(\mathcal{Z}_\alpha)}{\otimes}(-)$, we obtain a fully faithful functor

$$i_\sigma^*(\mathcal{A}_\alpha)\text{-mod}(\mathrm{Vect})\to \mathrm{Vect}.$$

This easily implies that either $\mathsf{e}\to i_\sigma^*(\mathcal{A}_\alpha)$ is an isomorphism, or $i_\sigma^*(\mathcal{A}_\alpha)=0$. The latter dichotomy implies that one of the above two possibilities for $\mathcal{A}_\alpha$ itself must hold.

[11]The algebra structure comes from the fact that $\mathbb{L}^{\mathrm{restr}}_{\check{G},\mathsf{k},\alpha}\circ \mathbb{L}^{\mathrm{restr},L}_{G,k,\alpha}$ is a $\mathrm{QCoh}(\mathcal{Z}_\alpha)$-linear monad.

3.2.9. We now consider the *reducible* case. Let $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}$ be a connected component whose closed point σ is a semi-simple local system. Then there exists a unique conjugacy class of Levi subgroups M (see [AGKRRV1, Sects. 3.6 and 3.7]) such that σ factors via an *irreducible* $\check{M}$-local system $\sigma_{\check{M}}$.

By induction on the semi-simple rank, we can assume that the assertion of Theorem 1.3.9(ii) is valid for M. Let $\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma_{\check{M}}}$ be the corresponding connected component of $\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k}}$.

We consider the following two cases of the behavior of the functor (3.4) with G replaced by M and $\mathcal{Z}_\alpha := \mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma_{\check{M}}}$:

(a) It is an equivalence;

(b) The left-hand side is zero.

3.2.10. We claim that in case (a), the functor (3.4) for G and $\mathcal{Z}_\alpha := \mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}$ is an equivalence.

Indeed, it is enough to show that the essential image of $\mathbb{L}^{\mathrm{restr}}_{\check{G},\mathsf{k},\alpha}$ generates the target category. Note that $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})$ is generated by the essential images of the functors

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})$$

for the parabolics in the given class of association.

Now the required assertion follows from Theorem 1.4.6 (for k).

3.2.11. We claim that in case (b), the left adjoint in (3.4) for G and $\mathcal{Z}_\alpha := \mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}$, is zero. This will show that the left-hand side is zero (by Corollary 3.2.6).

By construction and Property (B), we can identify the functor in question with

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}).$$

We will show that the above functor annihilates the generators. By Property (D), we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma_{\check{M}}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{K},\mathsf{k}}) & \xrightarrow{\mathrm{Eis}^-_!} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \\
\Big\downarrow{\scriptstyle \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & & \Big\downarrow{\scriptstyle \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} \\
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma_{\check{M}}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{M,\mathsf{k}}) & \xrightarrow[\mathrm{Eis}^-_!]{} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}).
\end{array}$$

We claim that the top horizontal arrows in these diagrams, taken for all the parabolics in the given class of association, generate the target. This follows from the combination of:

(i) The fact that $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{K}}$ is an equivalence;

(ii) Theorem 1.4.6 for K;

(iii) The corresponding fact on the spectral side.

Hence, it suffices to show that the anti-clockwise circuits in these diagrams vanish. However, this follows from the fact that the lower-left corner vanishes (this is the assumption in case (b)).

□[Theorem 1.3.9(i)]

3.2.12. *Proof of Theorem 1.3.9(iii).* We need to show that in the notations of (3.4), the category

$$\operatorname{QCoh}(\mathcal{Z}_\alpha) \underset{\operatorname{QCoh}(\operatorname{LS}^{\operatorname{restr}}_{GL_n})}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{GL_n}) \tag{3.7}$$

is non-zero for every connected component $\mathcal{Z}_\alpha$ of $\operatorname{LS}^{\operatorname{restr}}_{GL_n}$.

By induction, we may assume that the functor $\mathbb{L}^{\operatorname{restr}}_{GL_{n'}}$ is an equivalence for $n' < n$. In particular, it is an equivalence for proper Levi subgroups of $G = GL_n$. Hence, in the notations of Sect. 3.2.9 only scenario (a) occurs. In particular, the category (3.7) is non-zero whenever $\mathcal{Z}_\alpha$ corresponds to reducible local systems.

It remains to treat the case of $\mathcal{Z}_\alpha$ corresponding to an irreducible local system σ. It is enough to show that $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{GL_n})$ contains a non-zero Hecke eigensheaf corresponding to σ. However, this has been established in [FGV, Ga1].

□[Theorem 1.3.9(iii)]

3.3. **Proof of Proposition 3.2.4.**

3.3.1. Note that the functor ${}'\mathbb{L}^{\operatorname{restr},L}_{G,\mathsf{k}}$ preserves compactness (see Sect. 3.1.7). Hence, it is enough to show that for

$$\mathcal{F} \in \operatorname{IndCoh}_{\operatorname{Nilp}}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}})^c \text{ and } \mathcal{M} \in \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})^c,$$

there is a canonical isomorphism

$$\mathcal{H}om_{\operatorname{Shv}(\operatorname{Bun}_{G,\mathsf{k}})}({}'\mathbb{L}^{\operatorname{restr},L}_{G,\mathsf{k}}(\mathcal{F}), \mathcal{M}) \simeq \mathcal{H}om_{\operatorname{IndCoh}_{\operatorname{Nilp}}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}})}(\mathcal{F}, \mathbb{L}^{\operatorname{restr}}_{G,\mathsf{k}}(\mathcal{M})). \tag{3.8}$$

3.3.2. Consider the tempered quotients

$$\mathbf{u}^R : \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}}) \leftrightarrows \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})_{\operatorname{temp}} : \mathbf{u}$$

and

$$\mathbf{u}^{\operatorname{spec},R} : \operatorname{IndCoh}_{\operatorname{Nilp}}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}}) \leftrightarrows \operatorname{QCoh}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}}) : \mathbf{u}^{\operatorname{spec}}.$$

By construction, the functor $\mathbb{L}^{\operatorname{restr}}_{G,\mathsf{k}}$ induces a functor

$$\mathbb{L}^{\operatorname{restr}}_{G,\operatorname{temp},\mathsf{k}} : \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})_{\operatorname{temp}} \to \operatorname{QCoh}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}})$$

such that

$$\mathbb{L}^{\operatorname{restr}}_{G,\operatorname{temp},k} \circ \mathbf{u}^R \simeq \mathbb{L}^{\operatorname{restr}}_{G,\operatorname{coarse},\mathsf{k}} \simeq \mathbf{u}^{\operatorname{spec},R} \circ \mathbb{L}^{\operatorname{restr}}_{G,\mathsf{k}}.$$

In particular, the functor $\mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}}$ takes values in $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})_{\operatorname{temp}}$ (viewed as a subcategory) and provides a left adjoint to $\mathbb{L}^{\operatorname{restr}}_{G,\operatorname{temp},\mathsf{k}}$.

3.3.3. By Proposition 4.5.2 below, the functor ${}'\mathbb{L}^{\operatorname{restr},L}_{G,\mathsf{k}}$ induces a functor

$${}'\mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}} : \operatorname{QCoh}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}}) \to \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})_{\operatorname{temp}},$$

so that

$$\mathbf{u}^R \circ {}'\mathbb{L}^{\operatorname{restr},L}_{G,\mathsf{k}} \simeq {}'\mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}} \circ \mathbf{u}^{\operatorname{spec},R}.$$

From (3.3) we obtain that

$${}'\mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}} \simeq \mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}}.$$

In particular, we obtain that the functors

$$\mathbb{L}^{\operatorname{restr}}_{G,\operatorname{temp},\mathsf{k}} : \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_{G,\mathsf{k}})_{\operatorname{temp}} \leftrightarrows \operatorname{QCoh}(\operatorname{LS}^{\operatorname{restr}}_{\check{G},\mathsf{k}}) : {}'\mathbb{L}^{\operatorname{restr},L}_{G,\operatorname{temp},\mathsf{k}} \tag{3.9}$$

do form an adjoint pair.

3.3.4. Consider the maps

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})}({}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}(\mathcal{F}),\mathcal{M}) \to \mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{temp}}}(\mathbf{u}^R\circ {}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}(\mathcal{F}),\mathbf{u}^R(\mathcal{M})) = \tag{3.10}$$
$$= \mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{temp}}}({}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathrm{temp},k}\circ \mathbf{u}^{\mathrm{spec},R}(\mathcal{F}),\mathbf{u}^R(\mathcal{M})) \overset{\text{adjunction (3.9)}}{\simeq}$$
$$\simeq \mathcal{H}om_{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}(\mathbf{u}^{\mathrm{spec},R}(\mathcal{F}),\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{temp},\mathsf{k}}\circ \mathbf{u}^R(\mathcal{M})) =$$
$$= \mathcal{H}om_{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}(\mathbf{u}^{\mathrm{spec},R}(\mathcal{F}),\mathbf{u}^{\mathrm{spec},R}\circ \mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}(\mathcal{M})) \leftarrow \mathcal{H}om_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}(\mathcal{F},\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}(\mathcal{M}))$$

In order to establish (3.8), it suffices to show that the first and the last arrow in (3.10) are isomorphisms.

For the last arrow, this follows from the fact that the functor $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ sends compact objects to eventually coconnective objects (by construction), and the functor $\mathbf{u}^{\mathrm{spec},R}$ is fully faithful on the eventually coconnective subcategory.

For the first arrow, since $\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}(\mathcal{F})$ is compact (see Sect. 3.1.7), the assertion follows from the next lemma:

Lemma 3.3.5. *The restriction of the functor* $\mathbf{u}^R$ *to* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})^c$ *is fully faithful.*

Proof. Repeats verbatim the proof of [GLC1, Proposition 5.2.3].

□

□[Proposition 3.2.4]

3.4. **An alternative proof of Theorem 1.3.9(i) and Proposition 3.2.4.** This proof will use an additional property of the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$, given by Remark 7.4.3.

3.4.1. Let

$${}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}:\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$$

be defined as in (3.2).

We already know that this functor preserves compactness and by Property (A) of the functor (3.1), it is a Verdier quotient. I.e., the functor ${}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}$ admits a $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})$-linear fully faithful right adjoint. Denote it by ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ (note, however, that we do know yet that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ is isomorphic to $\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$).

We wish to show that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ is an equivalence onto a direct summand of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\mathsf{G},\mathsf{k}})$ corresponding to the union of some of the connected components of $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}$.

3.4.2. Inspecting the argument in Sects. 3.2.7-3.2.11, we see that the only place where we used that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}=\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ was in case (a) in Sect. 3.2.10. We provide an alternative argument as follows:

It suffices to prove that the functor ${}'\mathbb{L}^{\mathrm{restr},L}_{G,\mathsf{k}}|_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})}$ is conservative. In other words, we wish to show that in case (a) the functor

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}:\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})\to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}) \tag{3.11}$$

is conservative.

According to Remark 7.4.3, we have commutative diagrams

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{M,\mathsf{K},\mathsf{k}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{M,\mathsf{k}}) \\
{\scriptstyle \mathrm{CT}^-_*}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^-_*} \\
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k},\sigma})\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}})}{\otimes}\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}),
\end{array} \tag{3.12}$$

where CT^-_* denotes the constant term functor.

Now, the validity of GLC over K implies that the direct sums of the functors CT^-_* over the parabolics in the given class of association (i.e., the left vertical arrow in (3.12)) is conservative.

By induction on the semi-simple rank we can assume that the top horizontal arrow in (3.12) is conservative.

Hence, the bottom the top horizontal arrow in (3.12) is also conservative, as desired.

$\square$[Theorem 1.3.9(i)]

3.4.3. We now give an alternative argument for Proposition 3.2.4. Namely, we wish to show that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}} = \mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$. We will show that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ satisfies the two conditions of Corollary 1.3.6.

We already know that ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ induces an equivalence

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}({}'\,\mathrm{LS}^{\mathrm{restr}}_{\mathsf{k}}).$$

In particular, it sends compact objects to compact (and, in particular, eventually coconnective) ones. Hence, ${}'\mathbb{L}^{\mathrm{restr}}_{G,\mathsf{k}}$ satisfies condition (i) of Corollary 1.3.6.

Condition (ii) follows by passing to right adjoints from (3.3).

$\square$[Proposition 3.2.4]

4. Proof of Theorem 3.1.6

In this section we begin the proof of Theorem 3.1.6. Namely, we will construct the functor Sp and establish Properties (B)-(E).

Property (A) will be dealt with in the next section.

4.1. **Construction of the specialization functor.**

4.1.1. Let $\mathcal{Y}_{\mathsf{R}_0}$ be a scheme or algebraic stack over R_0. Our goal is to define a functor

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}(\mathcal{Y}_{\mathsf{K}}) \to \mathrm{Shv}(\mathcal{Y}_{\mathsf{k}}). \tag{4.1}$$

Denote

$$\mathrm{Spec}(\mathsf{k}) \xrightarrow{\mathbf{i}} \mathrm{Spec}(\mathsf{R}) \xleftarrow{\mathbf{j}} \mathrm{Spec}(\mathsf{K}).$$

We will use the same symbols $\mathbf{i}$ and $\mathbf{j}$ for the corresponding maps

$$\mathcal{Y}_{\mathsf{k}} \to \mathcal{Y}_{\mathsf{R}} \leftarrow \mathcal{Y}_{\mathsf{K}}.$$

Euphemistically, the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ is given by

$$\mathbf{i}^* \circ \mathbf{j}_*. \tag{4.2}$$

Note, however, that $\mathrm{Spec}(\mathsf{R})$ is non-Noetherian. In what follows we will rewrite the definition of $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ in terms of functors that only use algebro-geometric objects of finite type.

4.1.2. First, covering $\mathcal{Y}_{\mathsf{R}_0}$ smoothly be (affine) schemes, it is sufficient to define $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ for $\mathcal{Y}_{\mathsf{R}_0} = S_{\mathsf{R}_0}$, where S_{R_0} is an affine scheme of finite type over R_0, provided that it commutes with pullbacks along smooth maps $S'_{\mathsf{R}_0} \to S_{\mathsf{R}_0}$.

Second, since $\mathrm{Shv}(S_{\mathsf{K}})$ is by definition the ind-completion of $\mathrm{Shv}(S_{\mathsf{K}})^c$, it suffices to define $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ as a functor $\mathrm{Shv}(S_{\mathsf{K}})^c \to \mathrm{Shv}(S_{\mathsf{k}})^c$.

Third, by the definition of $\mathrm{Shv}(-)^c$, this category is the colimit of $\mathrm{Shv}(-)^c_E$, over $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$, where E is a finite extension of $\mathbb{Q}_\ell$. Hence, it suffices to define $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ as a functor $\mathrm{Shv}(S_{\mathsf{K}})^c_E \to \mathrm{Shv}(S_{\mathsf{k}})^c_E$.

Fourth, by the definition of $\mathrm{Shv}(-)^c_E$, it is obtained as the localization with respect to ℓ of $\mathrm{Shv}(-)^c_{\mathcal{O}_E}$. Hence, it suffices to define $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ as a functor $\mathrm{Shv}(S_{\mathsf{K}})^c_{\mathcal{O}_E} \to \mathrm{Shv}(S_{\mathsf{k}})^c_{\mathcal{O}_E}$.

Finally, by the definition of $\mathrm{Shv}(-)^c_{\mathcal{O}_E}$, it is obtained as

$$\lim_n \mathrm{Shv}(-)^c_{\mathcal{O}_E/\ell^n},$$

where the limit is taken in the ∞-category of *non-cocomplete* DG categorries.

Hence, it suffices to define $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ as a functor

$$\mathrm{Shv}(S_{\mathsf{K}})^c_{\mathcal{O}_E/\ell^n} \to \mathrm{Shv}(S_{\mathsf{k}})^c_{\mathcal{O}_E/\ell^n}. \tag{4.3}$$

4.1.3. Note that

$$\mathrm{Shv}(S_{\mathsf{K}})^c_{\mathcal{O}_E/\ell^n} = \underset{\mathsf{K}_0'}{\mathrm{colim}}\ \mathrm{Shv}(S_{\mathsf{K}_0'})^c_{\mathcal{O}_E/\ell^n},$$

where:

- The colimit is taken in the ∞-category of *non-cocomplete* DG categorries;
- The index category is the poset of finite extensions $\mathsf{K}_0' \supset \mathsf{K}_0$.

We proceed to define the corresponding functors

$$\mathrm{Sp}_{\mathsf{K}_0'\to\mathsf{k}} : \mathrm{Shv}(S_{\mathsf{K}_0'})^c_{\mathcal{O}_E/\ell^n} \to \mathrm{Shv}(S_{\mathsf{k}})^c_{\mathcal{O}_E/\ell^n}. \tag{4.4}$$

4.1.4. Let R_0' denote the integral closure of R_0 in K_0'. Consider the diagram

$$\begin{array}{ccccc} \mathcal{Y}_{\mathsf{k}} & \longrightarrow & \mathcal{Y}_{\mathsf{R}_0'} & \longleftarrow & \mathcal{Y}_{\mathsf{K}_0'} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathsf{k}) & \longrightarrow & \mathrm{Spec}(\mathsf{R}_0') & \longleftarrow & \mathrm{Spec}(\mathsf{K}_0'). \end{array}$$

We define (4.4) to be the nearby cycles functor Ψ.

4.1.5. Note that, by construction, the composition

$$\mathrm{Shv}(\mathcal{Y}_{\mathsf{K}_0}) \overset{\text{pullback}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}_{\mathsf{K}}) \overset{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}_{\mathsf{k}})$$

is the usual nearby cycles functor

$$\Psi : \mathrm{Shv}(\mathcal{Y}_{\mathsf{K}_0}) \to \mathrm{Shv}(\mathcal{Y}_{\mathsf{k}}).$$

4.1.6. We record the following properties of the functor (4.1):

(1) It is t-exact;

(2) It sends constructible[12] objects in $\mathrm{Shv}(\mathcal{Y}_{\mathsf{K}})$ to constructible objects in $\mathrm{Shv}(\mathcal{Y}_{\mathsf{K}})$;

(3) For $\mathcal{Y}_{\mathsf{R}_0} = \mathcal{Y}^1_{\mathsf{R}_0} \times \mathcal{Y}^2_{\mathsf{R}_0}$ and $\mathcal{F}_i \in \mathrm{Shv}(\mathcal{Y}^i_{\mathsf{K}})$, we have

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}_1) \boxtimes \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}_2). \tag{4.5}$$

(4) For a map $f : \mathcal{Y}^1_{\mathsf{R}_0} \to \mathcal{Y}^2_{\mathsf{R}_0}$ we have the natural transformations

$$(f_{\mathsf{k}})_! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (f_{\mathsf{K}})_!,\ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (f_{\mathsf{K}})_* \to (f_{\mathsf{k}})_* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}, \tag{4.6}$$

and

$$(f_{\mathsf{k}})^* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (f_{\mathsf{K}})^*,\ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (f_{\mathsf{K}})^! \to (f_{\mathsf{k}})^! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}, \tag{4.7}$$

where f_{k} (resp., f_{K}) denotes the fiber of f over $\mathrm{Spec}(\mathsf{k})$ (resp., $\mathrm{Spec}(\mathsf{K})$).

(5) The maps (4.6) are mutually inverse isomorphisms when f is proper and the maps (4.7) are mutually inverse isomorphisms (up to a shift by the relative dimension) when f is smooth.

---

[12]Constructible:=pullback to an affine scheme by means of a smooth morphism is constructible (i.e., compact). Note, however, that for stacks "constructible" does not imply "compact": the obstruction is (i) non-quasi-compactness of the stack and (ii) non-trivial stabilizers.

(6) It commutes with Verdier duality on constructible objects

$$\mathbb{D}_{\mathcal{Y}_{\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})) \simeq \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathcal{Y}_{\mathsf{K}}}(\mathcal{F})); \tag{4.8}$$

moreover, for an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y}_{\mathsf{K}})^{\mathrm{constr}}$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathbb{D}_{\mathcal{Y}_{\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})) & \xrightarrow[\mathrm{id}\boxtimes(4.8)]{\sim} & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathcal{Y}_{\mathsf{K}}}(\mathcal{F})) \\ \uparrow & & \uparrow \\ (\Delta_{\mathcal{Y}_{\mathsf{k}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}_{\mathsf{k}}}) & \longrightarrow & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathcal{Y}_{\mathsf{K}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}_{\mathsf{K}}}), \end{array} \tag{4.9}$$

where:

- The symbol $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ in the bottom-right corner refers to the functor (4.1) for $\mathcal{Y}_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} \mathcal{Y}_{\mathsf{R}_0}$;
- The bottom horizontal arrow comes from the identification $\underline{\mathsf{e}}_{\mathrm{Spec}(\mathsf{k})} \simeq \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\underline{\mathsf{e}}_{\mathrm{Spec}(\mathsf{K})})$ and the natural transformation

$$(\Delta_{\mathcal{Y}_{\mathsf{k}}})_! \circ \pi^*_{\mathcal{Y}_{\mathsf{k}}} \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathcal{Y}_{\mathsf{K}}})_! \circ \pi^*_{\mathcal{Y}_{\mathsf{K}}}$$

where $\pi_{\mathcal{Y}_{-}} : \mathcal{Y}_{-} \to \mathrm{Spec}(-)$ is the relevant structural map.

Remark 4.1.7. Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of constructible sheaves. From (4.9) we obtain a tautological commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{G}) \boxtimes \mathbb{D}_{\mathcal{Y}_{\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})) & \xrightarrow[\mathrm{id}\boxtimes(4.8)]{\sim} & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{G}) \boxtimes \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathcal{Y}_{\mathsf{K}}}(\mathcal{F})) \\ \uparrow & & \uparrow \\ (\Delta_{\mathcal{Y}_{\mathsf{k}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}_{\mathsf{k}}}) & \longrightarrow & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathcal{Y}_{\mathsf{K}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}_{\mathsf{K}}}). \end{array} \tag{4.10}$$

In this formulation, there is some additional homotopy coherence to note. Each term of (4.10) can now be considered as a functor from $\mathrm{TwArr}(\mathrm{Shv}(\mathcal{Y}_{\mathsf{K}}))^{\mathrm{constr}}$, the twisted arrow category of constructible sheaves, to $\mathrm{Shv}(\mathcal{Y}_{\mathsf{k}} \times_{\mathrm{Spec}(\mathsf{k})} \mathcal{Y}_{\mathsf{k}})$. Then the more functorial assertion of (6) is that there is an commutative square of functors (4.10).

4.1.8. The following observation will be used in the sequel:

Lemma 4.1.9. *Let $f : \mathcal{Y}_{\mathsf{R}_0} \to \mathcal{Z}_{\mathsf{R}_0}$ be a map of algebraic stacks over* $\mathrm{Spec}(\mathsf{R}_0)$*. Let $\mathcal{F}_{\mathcal{Y},\mathsf{R}_0} \in \mathrm{Shv}(\mathcal{Y}_{\mathsf{R}_0})$ be ULA over $\mathcal{Z}_{\mathsf{R}_0}$. Then for any $\mathcal{F}_{\mathcal{Z}} \in \mathrm{Shv}(\mathcal{Z}_{\mathsf{K}})$, the naturally defined map*

$$\mathcal{F}_{\mathcal{Y},\mathsf{k}} \overset{*}{\otimes} (f^*_{\mathsf{k}} \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{G}_{\mathcal{Z}})) \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}_{\mathcal{Y},\mathsf{K}} \overset{*}{\otimes} f^*_{\mathsf{K}}(\mathcal{F}_{\mathcal{Z}}))$$

*is a isomorphism, where $\mathcal{F}_{\mathcal{Y},\mathsf{k}}$ (resp., $\mathcal{F}_{\mathcal{Y},\mathsf{K}}$) denotes the *-restriction of $\mathcal{F}_{\mathcal{Y},\mathsf{R}_0}$ to $\mathcal{Y}_{\mathsf{k}}$ (resp., $\mathcal{Y}_{\mathsf{K}}$)*

Remark 4.1.10. Note that when $\mathcal{Z}_{\mathsf{R}_0} = \mathrm{Spec}(\mathsf{R}_0)$, the ULA condition simply means that

$$\Phi(\mathcal{F}_{\mathcal{Y},\mathsf{R}_0}) = 0,$$

where Φ is the vanishing cycles functor.

In other words, this is equivalent to the map

$$\mathcal{F}_{\mathcal{Y},\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}_{\mathcal{Y},\mathsf{K}})$$

being an isomorphism.

4.1.11. In what follows we will study the resulting functor

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}). \tag{4.11}$$

4.2. **Specialization and Hecke functors.**

4.2.1. Consider the version of the Hecke stack over $\operatorname{Spec}(\mathsf{R}_0)$:

$$\begin{array}{ccccc} \operatorname{Bun}_{G,\mathsf{R}_0} & \xleftarrow{\overleftarrow{h}} & \operatorname{Hecke}_{X,\mathsf{R}_0} & \xrightarrow{\overrightarrow{h}} & \operatorname{Bun}_{G,\mathsf{R}_0} \\ & & \Big\downarrow{\scriptstyle s} & & \\ & & X_{\mathsf{R}_0}. & & \end{array}$$

For an irreducible representation

$$V^\lambda \in \operatorname{Rep}(\check{G}), \quad \lambda \in \Lambda^+,$$

let

$$\operatorname{Sat}^{\mathrm{nv}}_{X,\mathsf{R}_0}(V^\lambda) \in \operatorname{Shv}(\operatorname{Hecke}_{X,\mathsf{R}_0})$$

be the object[13]

$$\operatorname{IC}_{\overline{\operatorname{Hecke}}^\lambda_{X,\mathsf{R}_0}}[-2].$$

We claim:

Theorem 4.2.2. *The object* $\operatorname{Sat}^{\mathrm{nv}}_{X,\mathsf{R}_0}(V^\lambda)$ *restricts to* $\operatorname{Sat}^{\mathrm{nv}}_{X,\mathsf{K}}(V^\lambda)$ *and* $\operatorname{Sat}^{\mathrm{nv}}_{X,\mathsf{k}}(V^\lambda)$*, respectively, and is ULA over* $\operatorname{Bun}_{G,\mathsf{R}_0} \underset{\operatorname{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$ *with respect to either of the projections.*

We will prove this theorem in Sect. 6.1.

4.2.3. For a finite set $\mathcal{I}$, consider the corresponding $\mathcal{I}$-legged Hecke stack:

$$\begin{array}{ccccc} \operatorname{Bun}_{G} & \xleftarrow{\overleftarrow{h}} & \operatorname{Hecke}_{X^{\mathcal{I}}} & \xrightarrow{\overrightarrow{h}} & \operatorname{Bun}_{G} \\ & & \Big\downarrow{\scriptstyle s} & & \\ & & X^{\mathcal{I}}. & & \end{array}$$

As in the usual geometric Satake theory, Theorem 4.2.2 allows us to construct a family of functors

$$\operatorname{Sat}^{\mathrm{nv}}_{X^{\mathcal{I}},\mathsf{R}_0} : \operatorname{Rep}(\check{G})^{\otimes \mathcal{I}} \to \operatorname{Shv}(\operatorname{Hecke}_{X^{\mathcal{I}},\mathsf{R}_0}). \tag{4.12}$$

Moreover, the essential image of the functor (4.12) lies in the full subcategory of $\operatorname{Shv}(\operatorname{Hecke}_{X^{\mathcal{I}},\mathsf{R}_0})$ that consists of objects that are ULA over $\operatorname{Bun}_{G,\mathsf{R}_0}$ (in fact, over $\operatorname{Bun}_{G,\mathsf{R}_0} \underset{\operatorname{Spec}(\mathsf{R}_0)}{\times} X^{\mathcal{I}}_{\mathsf{R}_0}$).

4.2.4. Consider the Hecke functor

$$\mathrm{H}_V : \operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Shv}(\operatorname{Bun}_G \times X^{\mathcal{I}}), \quad (\overleftarrow{h} \times s)_! \left(\overrightarrow{h}^*(-) \overset{*}{\otimes} \operatorname{Sat}^{\mathrm{nv}}_{X^{\mathcal{I}}}(V) \right), \quad V \in \operatorname{Rep}(\check{G})^{\otimes \mathcal{I}}.$$

The functors (4.12) give rise to natural transformations

$$\mathrm{H}_{V,\mathsf{k}} \circ \operatorname{Sp}_{\mathsf{K}\to\mathsf{k}} \to \operatorname{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \mathrm{H}_{V,\mathsf{K}} \tag{4.13}$$

as functors

$$\operatorname{Shv}(\operatorname{Bun}_{G,\mathsf{K}}) \to \operatorname{Shv}(\operatorname{Bun}_{G,\mathsf{k}} \times X^{\mathcal{I}}_{\mathsf{k}}).$$

We claim:

Proposition 4.2.5. *The natural transformations* (4.13) *are isomorphisms.*

[13] The cohomological shift $[-2]$ in the formula below is designed in order to offset $\dim(X) + \dim(\mathsf{R}_0)$.

Proof. The map
$$\overleftarrow{h} \times s : \mathrm{Hecke}_{X^{\mathcal{J}}} \to \mathrm{Bun}_G \times X^{\mathcal{J}}$$
is proper, hence
$$(\overleftarrow{h} \times s)_! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\overleftarrow{h} \times s)_!$$
is an isomorphism.

Now, the natural transformation
$$(\overrightarrow{h}^*(-) \overset{*}{\otimes} \mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}}}(V)) \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\overrightarrow{h}^*(-) \overset{*}{\otimes} \mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}}}(V))$$
is also an isomorphism by Lemma 4.1.9 and the ULA property of the objects $\mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}},\mathsf{R}_0}(V)$. □

4.2.6. The functors $\mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}},\mathsf{R}_0}$, $\mathcal{J} \in \mathrm{fSet}$ make the following system of diagrams of functors commute:

$$\begin{array}{ccc} \mathrm{Rep}(\check{G})^{\otimes \mathcal{J}_1} & \xrightarrow{\mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}_1},\mathsf{R}_0}} & \mathrm{Shv}(\mathrm{Hecke}_{X^{\mathcal{J}_1},\mathsf{R}_0}) \\ {\scriptstyle \mathrm{Rep}(\check{G})^{\phi}}\downarrow & & \downarrow {\scriptstyle \Delta^*_\phi} \\ \mathrm{Rep}(\check{G})^{\otimes \mathcal{J}_2} & \xrightarrow{\mathrm{Sat}^{\mathrm{nv}}_{X^{\mathcal{J}_2},\mathsf{R}_0}} & \mathrm{Shv}(\mathrm{Hecke}_{X^{\mathcal{J}_2},\mathsf{R}_0}), \end{array} \tag{4.14}$$

for $\phi : \mathcal{J}_1 \to \mathcal{J}_2$, where:

- The left vertical arrow is induced by the symmetric monoidal structure on $\mathrm{Rep}(\check{G})$;
- The map Δ_ϕ is (the base change of) the diagonal map $X^{\mathcal{J}_2} \to X^{\mathcal{J}_1}$.

This leads to a system of commutative diagrams of functors

$$\begin{array}{ccc} (\mathrm{id}_{\mathrm{Bun}_G} \times \Delta_\phi)^* \circ \mathrm{H}_{V,\mathsf{k}} \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} & \longrightarrow & (\mathrm{id}_{\mathrm{Bun}_G} \times \Delta_\phi)^* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \mathrm{H}_{V,\mathsf{K}} \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{H}_{\mathrm{Res}_\phi(V),\mathsf{k}} \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} & \longrightarrow & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \mathrm{H}_{\mathrm{Res}_\phi(V),\mathsf{K}}, \end{array}$$

which also depends functorially on $V \in \mathrm{Rep}(\check{G})^{\otimes \mathcal{J}}$.

4.3. **Compatibility with Beilinson's projector.**

4.3.1. Recall Beilinson's projector, denoted P, which is an idempotent acting on $\mathrm{Shv}(\mathrm{Bun}_G)$ with essential image $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, see [AGKRRV1, Sect. 13.4].

We will denote by P_{k} (resp., P_{K}) the corresponding endofunctor of $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})$ (resp., $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$).

Let now $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ be a version of P_{K}, where in we take $\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\mathsf{G},\mathsf{k}}, -)$ instead of $\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\mathsf{G},\mathsf{K}}, -)$. This is an idempotent endomorphism of $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})$, equal to the composition of P_{K} and the projection onto the direct summand
$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}}).$$

4.3.2. We claim:

Proposition 4.3.3. *There exists a canonical isomorphism*
$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \mathsf{P}_{\mathsf{K},\mathsf{k}} \simeq \mathsf{P}_{\mathsf{k}} \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}.$$

Proof. By construction (see [AGKRRV1, Sect. 13.1.11]), the functor $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ is the colimit of functors F_I, $I \in \mathrm{fSet}$, each of which is the composition of functors of the following form:

- (a) $\mathrm{H}_{V_I} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$ for $V_I \in \mathrm{Rep}(\check{G})^{\otimes I}$;
- (b) $(-) \otimes \mathrm{Ev}(V'_I) : \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \otimes \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$, where:
  - $V'_I \in \mathrm{Rep}(\check{G})^{\otimes I}$;
  - $\mathrm{Ev}(V'_I) \in \mathrm{QLisse}(X^I) \otimes \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ is the tautological object corresponding to V'_I.
- (c) $\mathrm{Id} \otimes \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}}, -)$.

The functors in (a) commute with $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ by Proposition 4.2.5. The functors in (b) commute with $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ by Lemma 4.1.9. The functors in (c) commute with $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ tautologically.

□

Remark 4.3.4. For future use we remark that both Proposition 4.2.5 and 4.3.3 remain valid if instead of $\mathrm{Bun}_{G,\mathsf{R}_0}$ we consider

$$\mathrm{Bun}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} \mathcal{Y}_{\mathsf{R}_0},$$

for some stack $\mathcal{Y}_{\mathsf{R}_0}$ over $\mathrm{Spec}(\mathsf{R}_0)$. The proofs remain the same.

4.3.5. From Proposition 4.3.3 we obtain:

Corollary 4.3.6. *The functor* (4.11) *sends*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$$

to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \subset \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}).$$

Remark 4.3.7. Note that Proposition 4.3.3 implies that the functor (4.11) sends all of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$ to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$, while killing the direct summands of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$ that are supported off

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{k}} \subset \mathrm{LS}^{\mathrm{restr}}_{\check{G},\mathsf{K}}.$$

This follows from the fact that the functor $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ applied to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}})$ acts as a projector on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$.

Remark 4.3.8. One can also prove that the functor (4.11) sends $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})$ as follows:

It follows from [AGKRRV1, Theorem 14.4.3] that the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

can be characterized as follows: it consists of those objects $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ for which for every $V \in \mathrm{Rep}(\check{G})$ the object

$$\mathrm{H}_V(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

belongs to the full subcategory[14]

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

Now, this property is preserved by the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ by (4.5).

4.3.9. Thus, thanks to Corollary 4.3.6 we obtain the functor (3.1).

Note that the functor (3.1) satisfies Property (E) from Sect. 3.1.4 holds. In fact, the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ is t-exact by Sect. 4.1.6.

4.3.10. We are now ready to establish Property (B) (from Sect. 3.1.4). Indeed, it follows from Proposition 4.2.5, since the spectral action is determined by the Hecke functors (see [AGKRRV1, Corollary 12.8.4(b)]).

4.4. **Properties (C) and (D) of the specialization functor.**

4.4.1. Note that the construction in [GLC1, Sect. 3.3] (see Sect. 7.1.1 below for a review) makes sense over R_0, and produces an object

$$\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0} \in \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{R}_0}).$$

In Sect. 7.1 we will prove:

Theorem 4.4.2. *The object* $\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}$ *is ULA over* $\mathrm{Spec}(\mathsf{R}_0)$.

[14]In [AGKRRV1, Theorem 14.4.3], this is formulated as belonging to $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{qLisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X)$; however, the apparently weaker condition of belonging to $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(X)$ is equivalent to this stronger condition: objects in the essential image of H_V are ULA over X, and any object in $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(X)$ that is ULA over X lies in $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{qLisse}(X)$.

4.4.3. By combining Theorem 4.4.2 with Proposition 4.3.3 and Lemma 4.1.9, we obtain Property (C).

4.4.4. Consider the stacks

$$\mathrm{Bun}_{P^-,\mathsf{R}_0} \overset{j}{\hookrightarrow} \widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0}$$

and the projection

$$\widetilde{\mathsf{q}}^- : \widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0} \to \mathrm{Bun}_{M,\mathsf{R}_0}.$$

In Sect. 6.3 we will prove:

Theorem 4.4.5. *The object*

$$j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-,\mathsf{R}_0}}) \in \mathrm{Shv}(\widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0})$$

is ULA with respect to the projection $\widetilde{\mathsf{q}}^-$.

4.4.6. By combining Theorem 4.4.5 and Lemma 4.1.9 we deduce Property (D) of (3.1).

4.5. **Specialization and temperedness.** In this subsection and formulate and prove a property of the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ that describes its interaction with the *temperization* functor.

4.5.1. We claim:

Proposition 4.5.2. *The functor*

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$$

induces a functor

$$\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{temp}} \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{temp}}$$

so that the diagram

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}) \\ {\scriptstyle \mathbf{u}^R}\downarrow & & \downarrow{\scriptstyle \mathbf{u}^R} \\ \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{temp}} & \longrightarrow & \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{temp}} \end{array}$$

commutes.

We will give two proofs of this proposition.

4.5.3. *First proof.* Let $x \in X$ be a chosen point (which we use to define $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}}$) and let $\mathrm{Sph}(G)_x$ is the spherical Hecke category at x (see [GLC2, Sect. 1.5]).

Recall (see [AG1, Sect. 18.4]) that the colocalization

$$\mathbf{u}^R : \mathrm{Shv}(\mathrm{Bun}_G) \leftrightarrows \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} : \mathbf{u}$$

can be characterized in terms of the action of the monoidal category

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt} \,/\, \mathrm{Ad}(\check{G}))$$

on $\mathrm{Shv}(\mathrm{Bun}_G)$, obtained as a composition of the *derived Satake* equivalence

$$\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x := \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt} \,/\, \mathrm{Ad}(\check{G})) \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}(G)_x$$

and the natural action of $\mathrm{Sph}(G)_x$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ by Hecke functors.

Thus, in order to prove Proposition 4.5.2, it suffices to show that the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ intertwines the actions of $\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x$ on $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})$ and on $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$, where we let x be a section $\mathrm{Spec}(\mathsf{R}_0) \to X_{\mathsf{R}_0}$.

By construction, the functor

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Sph}(G)_{x,\mathsf{K}} \to \mathrm{Sph}(G)_{x,\mathsf{k}}$$

intertwines the action of $\mathrm{Sph}(G)_{x,\mathsf{K}}$ on $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})$ with the action of $\mathrm{Sph}(G)_{x,\mathsf{k}}$ on $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})$.

Hence, it remains to show that the endomorphism ϕ of $\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x$ that makes the following diagram commute

$$\begin{array}{ccc} \mathrm{Sph}^{\mathrm{spec}}(\check{G})_x & \xrightarrow{\phi} & \mathrm{Sph}^{\mathrm{spec}}(\check{G})_x \\ {\scriptstyle \mathrm{Sat}_G}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \mathrm{Sat}_G} \\ \mathrm{Sph}(G)_{x,\mathsf{K}} & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Sph}(G)_{x,\mathsf{k}} \end{array}$$

is actually an automorphism.

We note that ϕ has the following properties:

- It is monoidal;
- It makes the diagram

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \xrightarrow{\mathrm{id}} & \mathrm{Rep}(\check{G}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sph}^{\mathrm{spec}}(\check{G})_x & \xrightarrow{\phi} & \mathrm{Sph}^{\mathrm{spec}}(\check{G})_x \end{array}$$

  commute (this follows from Theorem 4.2.2);
- It induces the identity map on endomorphisms of the unit object

$$\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \simeq \mathrm{End}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}(\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}) \overset{\phi}{\to} \mathrm{End}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}(\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}) \simeq \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}.$$

The third points follow from the fact that the composite

$$\mathrm{Sym}(\mathfrak{t}^*[-2])^W \simeq \mathrm{C}^\cdot(\mathrm{pt}/G_{\mathsf{K}}) \simeq \mathrm{End}_{\mathrm{Sph}(G)_{x,\mathsf{K}}}(\mathbf{1}_{\mathrm{Sph}(G)_{x,\mathsf{K}}}) \overset{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}}{\longrightarrow}$$
$$\to \mathrm{End}_{\mathrm{Sph}(G)_{x,\mathsf{K}}}(\mathbf{1}_{\mathrm{Sph}(G)_{x,\mathsf{k}}}) \simeq \mathrm{C}^\cdot(\mathrm{pt}/G_{\mathsf{k}}) \simeq \mathrm{Sym}(\mathfrak{t}^*[-2])^W$$

is the identity map, while the identification $\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \simeq \mathrm{End}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}(\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x})$ equals

$$\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \simeq \mathrm{Sym}(\check{\mathfrak{t}}[-2])^W \simeq$$
$$\simeq \mathrm{Sym}(\mathfrak{t}^*[-2])^W \simeq \mathrm{End}_{\mathrm{Sph}(G)_x}(\mathbf{1}_{\mathrm{Sph}(G)_x}) \overset{\mathrm{Sat}_G}{\simeq} \mathrm{End}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}(\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x}).$$

We now claim:

Lemma 4.5.4. *Any endomorphism of* $\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x$ *that has the above three properties is an automorphism.*

The proof of the lemma is given in Sect. 4.5.6 below.

□[First proof of Proposition 4.5.2]

4.5.5. *Second proof.* We have to show that the functor $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$ sends

$$\ker(\mathbf{u}^R) : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) \overset{\mathbf{u}^R}{\twoheadrightarrow} \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{temp}}$$

to

$$\ker(\mathbf{u}^R) : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}) \overset{\mathbf{u}^R}{\twoheadrightarrow} \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{temp}}.$$

Recall the characterization of the kernel of

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathbf{u}^R}{\to} \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{temp}}$$

in [FR, Sect. 4.3].

The required assertion follows from the fact that the functors involved in this characterization commute with $\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$.

□[Second proof of Proposition 4.5.2]

4.5.6. *Proof of Lemma 4.5.4.* We rewrite

$$\mathrm{Sph}^{\mathrm{spec}}(\check{G})_x \simeq \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}}.$$

Using the first two properties, we can de-equivariantize both sides, i.e., apply $\mathrm{Vect} \underset{\mathrm{Rep}(\check{G})}{\otimes} -$, and obtain a functor

$$\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod} \overset{\widetilde{\phi}}{\to} \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}$$

with the following properties:

- It is $\check{G}$-equivariant;
- It sends $\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \in \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}$ to itself;
- It makes the diagram

$$\begin{array}{ccc} \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} & \xrightarrow{\mathrm{id}} & \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \\ \downarrow & & \downarrow \\ \mathrm{Sym}(\check{\mathfrak{g}}[-2]) & & \mathrm{Sym}(\check{\mathfrak{g}}[-2]) \\ \sim\downarrow & & \sim\downarrow \\ \mathrm{End}_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}}(\mathrm{Sym}(\check{\mathfrak{g}}[-2])) & \xrightarrow{\widetilde{\phi}} & \mathrm{End}_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}}(\mathrm{Sym}(\check{\mathfrak{g}}[-2])) \end{array}$$

  commute.

It suffices to show that the bottom horizontal arrow in this diagram is an isomorphism. This arrow is a map of algebras, hence is determined by a map of vector spaces.

$$\check{\mathfrak{g}}[-2] \to \mathrm{Sym}(\check{\mathfrak{g}}[-2]).$$

The grading forces the latter map to come from a map of vector spaces $\check{\mathfrak{g}} \to \check{\mathfrak{g}}$. By $\check{G}$-equivariance, the latter map acts as a scalar on each simple factor.

However, the commutation of the above diagram forces this scalar to be 1.

□[Lemma 4.5.4]

5. Proof of Property (A)

The goal of this section is to establish Property (A) of the functor (3.1).

5.1. **The key input.**

5.1.1. Consider the object

$$(\Delta_{\mathrm{Bun}_{G,\mathsf{R}_0}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}}) \in \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} \mathrm{Bun}_{G,\mathsf{R}_0}). \tag{5.1}$$

5.1.2. The key input in the proof of Property (A) is provided by the following result:

Theorem 5.1.3. *The object* (5.1) *is ULA over* $\mathrm{Spec}(\mathsf{R}_0)$.

The theorem will be proved in Sect. 6.4. We now proceed with the proof of Property (A).

5.1.4. Combining Theorem 5.1.3 with Lemma 4.1.9, we obtain:

Corollary 5.1.5. *The natural map*

$$(\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{k}}}) \simeq (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{K}}}) \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{K}}})$$

is an isomorphism.

Using the commutation of specialization with Verdier duality, from Corollary 5.1.5 we obtain:

Corollary 5.1.6. *The canonical map*

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}) \to (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}) \simeq (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})$$

is an isomorphism.

5.2. Harder-Narasimhan strata.

5.2.1. Recall the notion of *contractive* locally closed substack, see [DG1, Sect. 5.2.1].

For $\theta \in \Lambda^+_{\mathbb{Q}}$, let

$$\mathrm{Bun}_G^{(\leq \theta)} \subset \mathrm{Bun}_G$$

be the open union of Harder-Narasimhan strata, as defined in [DG1, Sect. 7.3.4].

The main technical result of the paper [DG1], proved in Sect. 9.3 of *loc. cit.*, says that for all θ large enough and $\theta' \geq \theta$, the (locally) closed substack

$$\mathrm{Bun}_G^{(\leq \theta')} - \mathrm{Bun}_G^{(\leq \theta)} \subset \mathrm{Bun}_G^{(\leq \theta')}$$

is contractive.

The proof of this result applies equally well in the relative situation over $\mathrm{Spec}(\mathsf{R}_0)$.

5.2.2. Let $\theta \leq \theta'$ be as above and consider the corresponding open embedding

$$\jmath_{\theta,\theta'} : \mathrm{Bun}_G^{(\leq \theta)} \hookrightarrow \mathrm{Bun}_G^{(\leq \theta')}.$$

We are going to prove:

Proposition 5.2.3. *The natural transformations*

$$(\jmath_{\theta,\theta'})_! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\jmath_{\theta,\theta'})_!$$

and

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\jmath_{\theta,\theta'})_* \to (\jmath_{\theta,\theta'})_* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}$$

are isomorphisms.

Proof. We will prove the first isomorphism, as the second one is similar.

It is enough to prove the assertion after applying pullback with respect to a smooth surjective map. Hence, by the definition of contractiveness, it is enough to prove the more general Proposition 5.2.5 below.

□

5.2.4. Let $\mathcal{Y}_{\mathsf{R}_0}$ and $\mathcal{Z}_{\mathsf{R}_0}$ be as in [DG1, Sect. 5.1.1]. Denote

$$\mathcal{U}_{\mathsf{R}_0} := \mathcal{Y}_{\mathsf{R}_0} - \mathcal{Z}_{\mathsf{R}_0} \overset{\jmath}{\hookrightarrow} \mathcal{Y}_{\mathsf{R}_0}.$$

Proposition 5.2.5. *The natural transformation*

$$\jmath_! \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \jmath_!$$

is an isomorphism.

Proof. Let $\imath$ denote the closed embedding $\mathcal{Z}_{\mathsf{R}_0} \to \mathcal{Y}_{\mathsf{R}_0}$. It is enough to show that the natural transformation

$$\imath^* \circ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ \imath^*, \tag{5.2}$$

as functors $\mathrm{Shv}(\mathcal{Y}_{\mathsf{K}}) \to \mathrm{Shv}(\mathcal{Z}_{\mathsf{k}})$, is an isomorphism.

As in [DG1, Sects. 5.1.3-5.1.5], we can assume that[15] $\mathcal{Z} = \mathrm{pt}/\mathbb{G}_m \times Z$ and $\mathcal{Y} = \mathbb{A}^n/\mathbb{G}_m \times Z$ for some base Z, where $\mathbb{G}_m$ acts on $\mathbb{A}^n$ via the mth power of the standard character, where $m > 0$. Further, applying blow-up (see [DG1, Sect. 5.1.6]), we can assume that $n = 1$.

It is enough to prove that the map (5.2) is an isomorphism on the generators. We take the generators to be of the form

$$(f_m)_*(\mathcal{F}_{\mathbb{A}^1}) \boxtimes \mathcal{F}_Z,$$

where

- f_m is the "raising to the power m" map $\mathbb{A}^1 \to \mathbb{A}^1$;

[15]In the formulas below, $\mathrm{pt} := \mathrm{Spec}(\mathsf{R}_0)$, and similarly for $\mathbb{A}^1$ and $\mathbb{G}_m$.

- $\mathcal{F}_Z \in \mathrm{Shv}(Z)$,
- $\mathcal{F}_{\mathbb{A}^1} \in \mathrm{Shv}(\mathbb{A}^1)^{\mathbb{G}_m}$ is either $\delta_{0,\mathbb{A}^1}$ or $\mathsf{e}_{\mathbb{A}^1}$.

In the above cases, the fact that (5.2) is an isomorphism is evident.

□

5.3. **Specialization for the "co"-category.**

5.3.1. Recall the category $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}}$, see [AGKRRV2, Sects. 2.5 and C.2]. By definition,

$$\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}} \simeq \underset{\theta\in\Lambda^+_{\mathbb{Q}}}{\mathrm{colim}}\, \mathrm{Shv}(\mathrm{Bun}_G^{(\leq\theta)}),$$

where the colimit is taken with respect to the functors $(\jmath_{\theta,\theta'})_*$.

From Proposition 5.2.3 we obtain that there exists a well-defined functor

$$\mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{co}} \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}} \tag{5.3}$$

that makes the following diagrams commute

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{co}} & \xrightarrow{\mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}} \\ {\scriptstyle (\jmath_\theta)_{\mathrm{co},*}}\uparrow & & \uparrow{\scriptstyle (\jmath_\theta)_{\mathrm{co},*}} \\ \mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}), \end{array} \tag{5.4}$$

where:

- $\jmath_\theta$ denotes the open embedding $\mathrm{Bun}_G^{(\leq\theta)} \hookrightarrow \mathrm{Bun}_G$;
- $(\jmath_\theta)_{\mathrm{co},*}$ denotes the corresponding tautological functor $\mathrm{Shv}(\mathrm{Bun}_G^{(\leq\theta)}) \to \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}}$ (not to be confused with $(\jmath_\theta)_* : \mathrm{Shv}(\mathrm{Bun}_G^{(\leq\theta)}) \to \mathrm{Shv}(\mathrm{Bun}_G)$.

5.3.2. Moreover, the following diagram commutes tautologically

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{co}} & \xrightarrow{\mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}} \\ \downarrow{\scriptstyle \mathrm{Ps\text{-}Id}^{\mathrm{nv}}} & & \downarrow{\scriptstyle \mathrm{Ps\text{-}Id}^{\mathrm{nv}}} \\ \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}) & \xrightarrow{\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}} & \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}), \end{array} \tag{5.5}$$

where

$$\mathrm{Ps\text{-}Id}^{\mathrm{nv}} : \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{Shv}(\mathrm{Bun}_G)$$

is as in [AGKRRV2, Sect. C.2.3].

5.3.3. Recall that for $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)^c$, its Verdier dual is well-defined as an object

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}}.$$

The commutation of specialization with Verdier duality implies that for $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}})^c$ we have a canonical isomorphism

$$\mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) \simeq \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})) \tag{5.6}$$

as objects in $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}$.

5.3.4. We now consider the "mixed" category

$$\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2},$$

defined in [AGKRRV2, Sect. C.4.2].

By definition,

$$\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2} = \underset{\theta\in\Lambda^+_{\mathbb{Q}}}{\mathrm{colim}}\, \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}),$$

where the colimit is taken with respect to the functors $(\mathrm{id} \times \jmath_{\theta,\theta'})_*$.

According to [AGKRRV2, Sect. C.4.3], the functor

$$\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2} \to \underset{\mathcal{U}}{\lim}\, \mathrm{Shv}(\mathcal{U} \times \mathrm{Bun}_G)_{\mathrm{co}} \tag{5.7}$$

is an equivalence, where:

- $\mathcal{U}$ runs over the poset of quasi-compact open substacks of Bun_G;
- For $\mathcal{U} \subset \mathcal{U}'$, the functor $\mathrm{Shv}(\mathcal{U}' \times \mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{Shv}(\mathcal{U} \times \mathrm{Bun}_G)_{\mathrm{co}}$ is given by restriction.

The discussion in Sect. 5.3.1 applies equally to the mixed situation.

In addition, we have a tautologically defined functor

$$\text{Ps-Id}^{\mathrm{nv}} : \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2} \to \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

and the corresponding counterpart of diagram (5.5) commutes.

5.3.5. We will consider the corresponding functor

$$\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}} \times \mathrm{Bun}_{G,\mathsf{K}})_{\mathrm{co}_2} \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}_2}.$$

5.3.6. Recall now the object

$$(\Delta_{\mathrm{Bun}_G})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_G}) \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2},$$

defined in [AGKRRV2, Sect. C.4.6].

In terms of the equivalence (5.7), the restriction of $(\Delta_{\mathrm{Bun}_G})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_G})$ to a given $\mathcal{U} \times \mathrm{Bun}_G$ is

$$(\mathrm{id} \times \jmath)_{\mathrm{co},*} \circ (\Delta_{\mathcal{U}})_*(\omega_{\mathcal{U}}), \quad \jmath : \mathcal{U} \hookrightarrow \mathrm{Bun}_G\,.$$

We have

$$\text{Ps-Id}^{\mathrm{nv}}((\Delta_{\mathrm{Bun}_G})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_G})) \simeq (\Delta_{\mathrm{Bun}_G})_*(\omega_{\mathrm{Bun}_G}),$$

as objects of $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$.

5.3.7. We are going to prove:

Proposition 5.3.8. *There exist a unique isomorphism*

$$\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right) \simeq (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}}) \tag{5.8}$$

as objects of $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}_2}$ *that makes the diagram*

$$\begin{array}{ccc}
\text{Ps-Id}^{\mathrm{nv}} \circ \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right) & \xrightarrow[(5.8)]{\sim} & \text{Ps-Id}^{\mathrm{nv}} \circ (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}}) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \circ (\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}) & \xrightarrow[\text{Corollary 5.1.6}]{\sim} & (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})
\end{array}$$

commute.

Proof. Using the equivalence (5.7), we have to show that there exists a unique family of isomorphisms

$$\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right)|_{\mathcal{U}_\mathsf{k}\times\mathrm{Bun}_{G,\mathsf{k}}} \simeq (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})|_{\mathcal{U}_\mathsf{k}\times\mathrm{Bun}_{G,\mathsf{k}}} \tag{5.9}$$

as objects of $\mathrm{Shv}(\mathcal{U}_\mathsf{k}\times\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}$, compatible via $\text{Ps-Id}^{\mathrm{nv}}$ with the isomorphism

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}\circ(\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})|_{\mathcal{U}_\mathsf{k}\times\mathrm{Bun}_{G,\mathsf{k}}} \simeq (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})|_{\mathcal{U}_\mathsf{k}\times\mathrm{Bun}_{G,\mathsf{k}}},$$

induced by the isomorphism of Corollary 5.1.6, where $\mathcal{U}$ runs over the poset of quasi-compact open substacks of Bun_G.

We can replace the poset of all $\mathcal{U}$ by a cofinal one that consists of the substacks $\mathrm{Bun}_G^{(\leq\theta)}$ for θ sufficiently large.

The left-hand side in (5.9) is by definition

$$\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\circ(\mathrm{id}\times\jmath_\theta)_{\mathrm{co},*}\circ(\Delta_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}}). \tag{5.10}$$

Using (5.4), we write it as

$$(\mathrm{id}\times\jmath_\theta)_{\mathrm{co},*}\left(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}\circ(\Delta_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{K}}})\right). \tag{5.11}$$

The right-hand in (5.9) is

$$(\mathrm{id}\times\jmath_\theta)_{\mathrm{co},*}\left((\Delta_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}})_*(\omega_{\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}})\right). \tag{5.12}$$

By Corollary 5.1.6, the expressions in (5.11) and (5.12) become isomorphic after applying the functor

$$\text{Ps-Id}^{\mathrm{nv}}:\mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}\times\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}\to\mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}\times\mathrm{Bun}_{G,\mathsf{k}}).$$

Now, the required assertion follows from the fact that the functor $\text{Ps-Id}^{\mathrm{nv}}$ is fully faithful on the essential image of

$$(\mathrm{id}\times\jmath_\theta)_{\mathrm{co},*}:\mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}\times\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}})\to\mathrm{Shv}(\mathrm{Bun}^{(\leq\theta)}_{G,\mathsf{k}}\times\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}.$$

□

5.4. **Method of proof.** In this subsection we launch the proof of Property (A) proper.

5.4.1. The proof is based on the following principle:

Let $F:\mathbf{C}_1\to\mathbf{C}_2$ be a functor between compactly generated categories. Assume that F preserves compactness, so it admits a continuous right adjoint, denoted F^R. Denote

$$F^{\mathrm{op}}:=(F^R)^\vee,\quad \mathbf{C}_1^\vee\to\mathbf{C}_2^\vee.$$

Explicitly, identifying

$$\mathbf{C}_i^\vee:=\mathrm{Ind}((\mathbf{C}_i^c)^{\mathrm{op}}),$$

the functor F^{op} is obtained by ind-extending the same-named functor on compact objects.

5.4.2. Consider the tautological map

$$(F\otimes F^{\mathrm{op}})(\mathbf{u}_{\mathbf{C}_1})\to\mathbf{u}_{\mathbf{C}_2}, \tag{5.13}$$

where $\mathbf{u}_{\mathbf{C}_i}\in\mathbf{C}_i\otimes\mathbf{C}_i^\vee$ is the unit of the self-duality.

The map (5.13) is characterized as follows. For $\mathbf{c}_1\in\mathbf{C}_1^c$ with formal dual $\mathbf{c}_1^\vee\in\mathbf{C}_1^\vee$, the diagram

$$\begin{array}{ccc}(F\otimes F^{\mathrm{op}})(\mathbf{u}_{\mathbf{C}_1}) & \xrightarrow{(5.13)} & \mathbf{u}_{\mathbf{C}_2}\\ \uparrow & & \uparrow\\ (F\otimes F^{\mathrm{op}})(\mathbf{c}_1\otimes\mathbf{c}_1^\vee) & \xrightarrow{\sim} & F(\mathbf{c}_1)\otimes F(\mathbf{c}_1)^\vee,\end{array} \tag{5.14}$$

commutes, where the vertical arrows are the canonical maps

$$\mathrm{can}_{\mathbf{c}}:\mathbf{c}\otimes\mathbf{c}^\vee\to u_{\mathbf{C}},\quad \mathbf{c}\in\mathbf{C}^c \tag{5.15}$$

for a compactly generated category $\mathbf{C}$.

Remark 5.4.3. Properly speaking, the object $u_{\mathbf{C}} \in \mathbf{C} \otimes \mathbf{C}^\vee$ can be characterized as the colimit over the the category of twisted arrows in $\mathbf{C}^c$ that sends

$$(\phi : \mathbf{c}'' \to \mathbf{c}') \in \mathrm{TwArr}(\mathbf{C}^c) \mapsto (\mathbf{c}' \otimes (\mathbf{c}'')^\vee) \in \mathbf{C} \otimes \mathbf{C}^\vee.$$

Indeed, for $\phi : \mathbf{c}'' \to \mathbf{c}'$ as above, the corresponding map $\mathbf{c}' \otimes (\mathbf{c}'')^\vee \to u_{\mathbf{C}}$ is either of the circuits in the following commutative diagram

$$\begin{array}{ccc} \mathbf{c}' \otimes (\mathbf{c}'')^\vee & \xrightarrow{\mathrm{id} \otimes \phi^\vee} & \mathbf{c}' \otimes (\mathbf{c}')^\vee \\ {\scriptstyle \phi \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{can}_{\mathbf{c}'}} \\ \mathbf{c}'' \otimes (\mathbf{c}'')^\vee & \xrightarrow{\mathrm{can}_{\mathbf{c}''}} & u_{\mathbf{C}}. \end{array}$$

In what follows, we have chosen to simplify the exposition by not explicitly mentioning the twisted arrows functoriality and working only with a single object at a time. To make the discussion complete, one replaces the eventual reference to (4.9) with a reference to Remark 4.1.7.

5.4.4. We have:

Lemma 5.4.5. *The functor F is a Verdier quotient if and only if the map* (5.13) *is an isomorphism.*

Proof. We need to show that the counit of the adjunction

$$F \circ F^R \to \mathrm{Id}$$

is an isomorphism. I.e., we have to show that the map

$$((F \circ F^R) \otimes \mathrm{Id})(\mathbf{u}_{\mathbf{C}_2}) \to \mathbf{u}_{\mathbf{C}_2} \tag{5.16}$$

is an isomorphism.

Note that F^R identifies with the functor dual to F^{op}. Hence, we can rewrite

$$((F \circ F^R) \otimes \mathrm{Id})(\mathbf{u}_{\mathbf{C}_2}) = (F \otimes \mathrm{Id}) \circ (F^R \otimes \mathrm{Id}) \circ (\mathbf{u}_{\mathbf{C}_2}) \simeq (F \otimes \mathrm{Id}) \circ ((F^{\mathrm{op}})^\vee \otimes \mathrm{Id}) \circ (\mathbf{u}_{\mathbf{C}_2}) \simeq$$
$$\simeq (F \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes F^{\mathrm{op}})(\mathbf{u}_{\mathbf{C}_1}) = (F \otimes F^{\mathrm{op}})(\mathbf{u}_{\mathbf{C}_1}).$$

Under this identification, the map (5.16) becomes the map (5.13).

□

5.4.6. We will show that the situation described above takes place for

$$\mathbf{C}_1 := \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K}}),\ \mathbf{C}_1 := \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}),\ F = \mathrm{Sp}_{\mathsf{K} \to \mathsf{k}}.$$

5.4.7. First, recall that the functor (3.1) preserves compactness, see Sect. 3.1.7 (this relies on Properties (B), (C) and (D) , which are proved independently of Property (A)).

Next, recall that (when working over a field), according to [AGKRRV2, Sects. 2.5.8, Corollary 2.6.5 and Proposition 2.7.6], the dual of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ identifies with

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{co}} \subset \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}}$$

in such a way that the unit of the duality is the object

$$\mathsf{P}((\Delta_{\mathrm{Bun}_G})_*^{\mathrm{fine}}(\omega_{\mathrm{Bun}_G})), \tag{5.17}$$

where:

- P denotes here Beilinson's projector, applied along the first factor;
- The object in (5.17), which a priori lies in $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2}$ belongs in fact to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{co}} \subset \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2}.$$

5.4.8. Furthermore (still working over a field), the corresponding equivalence

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^c)^{\mathrm{op}} \simeq (\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{co}})^c$$

is induced by the Verdier duality equivalence

$$(\mathrm{Shv}(\mathrm{Bun}_G)^c)^{\mathrm{op}} \simeq (\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{co}})^c,$$

where we are using the fact that the embedidng

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)^c$$

preserves compactness, by Theorem 1.1.7.

5.4.9. A variant of the above discussion applies to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})$, where we only need to replace P by $\mathsf{P}_{\mathsf{K},\mathsf{k}}$.

5.4.10. We claim that the functor $\mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}$ of (5.3) sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})_{\mathrm{co}} \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}$$

and identifies with $(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}})^{\mathrm{op}}$ (in the notations of Sect. 5.4.1).

Indeed, this follows by combining (5.6) and Sect. 5.4.8.

5.4.11. Thus, by Lemma 5.4.5 it remains to show that the map (5.13), which in our case is the map

$$(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}} \otimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}) \left(\mathsf{P}_{\mathsf{K},\mathsf{k}}((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}))\right) \to \mathsf{P}_{\mathsf{k}}((\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})), \tag{5.18}$$

is an isomorphism in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}}$.

5.5. **Verification.**

5.5.1. Applying the (fully faithful) functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}} \to \\ \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}}) \otimes \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}} \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}_2},$$

it suffices to show that (5.18) is an isomorphism of objects in $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}_2}$.

5.5.2. By Proposition 4.3.3, we identify the left-hand side in (5.18) with

$$\mathsf{P}_{\mathsf{k}} \circ \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}} \left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right),$$

and using Proposition 5.3.8, we identify it further with

$$\mathsf{P}_{\mathsf{k}} \left((\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})\right),$$

which is the right-hand side in (5.18).

It remains to show that the map in (5.18) corresponds under the above identifications to the identity map.

5.5.3. By (5.14), we have to show that for $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}})^c$, the isomorphism

$$\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left(\mathsf{P}_{\mathsf{K},\mathsf{k}}((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}))\right) \overset{\text{Proposition 4.3.3}}{\simeq} \simeq \mathsf{P}_{\mathsf{k}} \circ \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right) \overset{\text{Proposition 5.3.8}}{\simeq} \mathsf{P}_{\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})\right) \tag{5.19}$$

makes the following diagram commute

$$\begin{array}{ccc} \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left(\mathsf{P}_{\mathsf{K},\mathsf{k}}((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}))\right) & \xrightarrow{(5.19)} & \mathsf{P}_{\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}})\right) \\ \uparrow & & \uparrow \\ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) & \xrightarrow{\sim} & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})), \end{array} \tag{5.20}$$

where:

- The left vertical arrow is

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) \overset{(4.5)}{\simeq} \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F} \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) \to \to \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}\left(\mathsf{P}^{\mathrm{co}_2}_{\mathsf{K},\mathsf{k}}((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}))\right), \tag{5.21}$$

  where the last arrow is obtained by applying $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ to the canonical map (5.15), which in our case is

$$\mathcal{F} \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F}) \to (\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}); \tag{5.22}$$

- The right vertical arrow is the map (5.15).

5.5.4. Recall now that due to the validity of Theorem 1.1.7, the functor P is the right adjoint of the embedding emb. Nilp, see [AGKRRV1, Proposition 17.2.3].

From here, we formally obtain that the functor $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ provides a right adjoint to the embedding

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,\mathsf{K},\mathsf{k}}) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{K}}).$$

We obtain that the commutation of diagram (5.20) is equivalent to the commutation of the following diagram:

$$\begin{array}{ccc} \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right) & \xrightarrow{\text{Proposition 5.3.8}} & (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}}) \\ \uparrow & & \uparrow \\ \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) & \xrightarrow{\sim} & \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F})), \end{array} \tag{5.23}$$

where:

- The left vertical arrow is

$$\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) \overset{(4.5)}{\simeq} \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F} \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) \to \to \mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}(\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})^{\mathrm{fine}}_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}}), \tag{5.24}$$

  where the last arrow is obtained by applying $\mathsf{P}_{\mathsf{K},\mathsf{k}}$ to the canonical map (5.26) below;

- The right vertical arrow is the map (5.26).

5.5.5. Let $\mathcal{Y}$ be an algebraic stack. For an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$ there exists a canonical map

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \to \mathcal{F} \boxtimes \mathbb{D}_{\mathcal{Y}}(\mathcal{F}).$$

Passing to Verdier duals, we obtain a map

$$\mathcal{F} \boxtimes \mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \to (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}). \tag{5.25}$$

5.5.6. We take $\mathcal{Y} = \mathrm{Bun}_G$ and $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)^c$. In this case, the map (5.25) lifts canonically to a map

$$\mathcal{F} \boxtimes \mathbb{D}_{\mathrm{Bun}_G}(\mathcal{F}) \to (\Delta_{\mathrm{Bun}_G})_*^{\mathrm{fine}}(\omega_{\mathrm{Bun}_G}) \tag{5.26}$$

in $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2}$.

Indeed, both sides in (5.26) belong to the essential image of

$$(\mathrm{id} \times \jmath)_{\mathrm{co},*} : \mathrm{Shv}(\mathrm{Bun}_G \times \mathcal{U}) \to \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_2}$$

for a quasi-compact $\mathcal{U} \overset{\jmath}{\hookrightarrow} \mathrm{Bun}_G$, cf. proof of Proposition 5.3.8.

5.5.7. Now, it follows by Verdier duality from (4.9) that the diagram

$$\begin{CD}
\mathrm{Sp}^{\mathrm{co}_2}_{\mathsf{K}\to\mathsf{k}}\left((\Delta_{\mathrm{Bun}_{G,\mathsf{K}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{K}}})\right) @>{\text{Proposition 5.3.8}}>> (\Delta_{\mathrm{Bun}_{G,\mathsf{k}}})_*(\omega_{\mathrm{Bun}_{G,\mathsf{k}}}) \\
@AAA @AAA \\
\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathrm{Sp}^{\mathrm{co}}_{\mathsf{K}\to\mathsf{k}}(\mathbb{D}_{\mathrm{Bun}_{G,\mathsf{K}}}(\mathcal{F})) @>{\sim}>> \mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}) \boxtimes \mathbb{D}_{\mathrm{Bun}_{G,\mathsf{k}}}(\mathrm{Sp}_{\mathsf{K}\to\mathsf{k}}(\mathcal{F}))
\end{CD} \tag{5.27}$$

obtained from (5.23) by applying

$$\mathrm{Ps}\text{-}\mathrm{Id}^{\mathrm{nv}} : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})_{\mathrm{co}_2} \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{k}} \times \mathrm{Bun}_{G,\mathsf{k}})$$

does commute.

This formally implies the commutation of (5.23) by the same principle as in the proof of Proposition 5.3.8.

6. Proofs of the local acyclicity theorems

The goal of this section is to prove the ULA theorems stated in the previous section.

6.1. **Proof of Theorem 4.2.2.**

6.1.1. Let $\Delta^\lambda_{X,\mathsf{R}_0}, \nabla^\lambda_{X,\mathsf{R}_0} \in \mathrm{Shv}(\mathrm{Hecke}_{X,\mathsf{R}_0})$ be the standard and costandard objects corresponding to λ, respectively. I.e., they are, respectively, the !- and *- extensions of the (cohomologically shifted) constant sheaf on $\mathrm{Hecke}^\lambda_{X,\mathsf{R}_0}$.

It is enough to show that $\Delta^\lambda_{X,\mathsf{R}_0}$ and $\nabla^\lambda_{X,\mathsf{R}_0}$ are ULA over $\mathrm{Bun}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$. Indeed, this would imply that $\mathrm{IC}^\lambda_{X,\mathsf{R}_0} := \mathrm{IC}_{\overline{\mathrm{Hecke}}^\lambda_{X,\mathsf{R}_0}}$ is also ULA and has the specified restrictions over k (this follows from the fact that the ULA condition is inherited by the passage to subquotients of perverse cohomologies, see [HS, Corollary 1.12]).

We will prove the assertion for $\Delta^\lambda_{X,\mathsf{R}_0}$; the assertion for $\nabla^\lambda_{X,\mathsf{R}_0}$ will follow by duality.

6.1.2. Let $\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0}$ be the moduli stack whose S-points are:

- An S-point x of X_{R_0};
- A G-bundle $\mathcal{P}$ on $X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S$;
- A reduction to B of the restriction of $\mathcal{P}$ along $S \overset{(x,\mathrm{id})}{\to} X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S$.

Consider the fiber product

$$'\mathrm{Hecke}_{X,\mathsf{R}_0} := \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Bun}_{G,\mathsf{R}_0}}{\times} \mathrm{Hecke}_{X,\mathsf{R}_0},$$

equipped with the maps

$$\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \overset{'\overleftarrow{h}}{\leftarrow} {'\mathrm{Hecke}_{X,\mathsf{R}_0}} \overset{'\overrightarrow{h}}{\to} \mathrm{Bun}_{G,\mathsf{R}_0}.$$

It is naturally stratified by locally closed subsets

$$'\mathrm{Hecke}^{\lambda'}_{X,\mathsf{R}_0}, \quad \lambda' \in \Lambda \simeq W\backslash W^{\mathrm{aff,ext}},$$

where $W^{\mathrm{aff,ext}}$ is the extended affine Weyl group. Denote by $'\Delta^{\lambda'}_{X,\mathsf{R}_0}$ the corresponding standard object.

The pullback of $\Delta^{\lambda}_{X,\mathsf{R}_0}$ along the (smooth) projection

$$'\mathrm{Hecke}_{X,\mathsf{R}_0} \to \mathrm{Hecke}_{X,\mathsf{R}_0}$$

admits a filtration with subquotients $'\Delta^{\lambda'}_{X,\mathsf{R}_0}$ for $\lambda' \in \Lambda$ projecting to $\lambda \in \Lambda^+ \simeq \Lambda/W$.

Hence, order to prove that $\Delta^{\lambda}_{X,\mathsf{R}_0}$ is ULA over $\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0}$ (with respect to both $\overleftarrow{h}$ and $\overrightarrow{h}$), it suffices to show that the objects $'\Delta^{\lambda'}_{X,\mathsf{R}_0}$ are ULA over $\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0}$ with respect to $'\overleftarrow{h}$ and over $\mathrm{Bun}_{G,\mathsf{R}_0}$ with respect to $'\overrightarrow{h}$.

6.1.3. Let $\mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0}$ be the moduli stack[16] whose S-points are:

- An S-point x of X_{R_0};
- A pair of G-bundles $\mathcal{P}$ and $\mathcal{P}'$ on $X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S$;
- Reductions to B of the restrictions of $\mathcal{P}$ and $\mathcal{P}'$ along $S \overset{(x,\mathrm{id})}{\to} X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S$;
- An isomorphism $\mathcal{P}|_{X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S - S} \simeq \mathcal{P}'|_{X_{\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} S - S}$.

Denote by $\overleftarrow{h}^{\mathrm{Fl}}$ and $\overrightarrow{h}^{\mathrm{Fl}}$ the natural projections

$$\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \overset{\overleftarrow{h}^{\mathrm{Fl}}}{\leftarrow} \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \overset{\overrightarrow{h}^{\mathrm{Fl}}}{\to} \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0}.$$

The prestack $\mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0}$ is an ind-algebraic stack. It is naturally stratified by locally closed substacks $\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0}$, where $\widetilde{w}$ runs over $W^{\mathrm{aff,ext}}$.

For a given $\widetilde{w} \in \widetilde{W}$, let

$$\Delta^{\widetilde{w}}_{X,\mathsf{R}_0} \in \mathrm{Shv}(\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0})$$

denote the corresponding standard object (i.e., the !-extension of the constant sheaf on $\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0}$).

6.1.4. We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} & \longrightarrow & '\mathrm{Hecke}_{X,\mathsf{R}_0}) \\ {\scriptstyle \overrightarrow{h}^{\mathrm{Fl}}}\downarrow & & \downarrow{\scriptstyle '\overrightarrow{h}} \\ \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} & \longrightarrow & \mathrm{Bun}_{G,\mathsf{R}_0} \end{array}$$

with the horizontal maps being smooth and proper.

The object $'\Delta^{\lambda'}_{X,\mathsf{R}_0} \in \mathrm{Shv}('\mathrm{Hecke}_{X,\mathsf{R}_0}))$ is isomorphic (up to a cohomological shift) to the direct image of $\Delta^{\widetilde{w}}_{X,\mathsf{R}_0} \in \mathrm{Shv}(\mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0})$ for any $\widetilde{w} \in W^{\mathrm{aff,ext}}$ that projects to λ' under $W^{\mathrm{aff,ext}} \to W\backslash W^{\mathrm{aff,ext}} \simeq \Lambda$.

Hence, it is enough to show that the objects $\Delta^{\widetilde{w}}_{X,\mathsf{R}_0}$ are ULA over $\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$ for both $\overleftarrow{h}^{\mathrm{Fl}}$ and $\overrightarrow{h}^{\mathrm{Fl}}$.

[16]Here "Fl" stands for "affine flags", as opposed to the usual Hecke stack, which is modeled on the affine Grassmannian.

6.1.5. We argue by induction on the length $\ell(\widetilde{w})$.

When $\ell(\widetilde{w})=0$, the map

$$\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0} \overset{\overleftarrow{h}\times s \text{ or } \overrightarrow{h}\times s}{\longrightarrow} \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$$

is an isomorphism, and there is nothing to prove.

When $\ell(\widetilde{w})=1$, the map

$$\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0} \overset{\overleftarrow{h}\times s \text{ or } \overrightarrow{h}\times s}{\longrightarrow} \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$$

is smooth fibration with fibers isomorphic to $\mathbb{A}^1$. Similarly, the closure

$$\overline{\mathrm{Hecke}}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0} \supset \mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0}$$

is a smooth fibration

$$\overline{\mathrm{Hecke}}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0} \overset{\overleftarrow{h}\times s \text{ or } \overrightarrow{h}\times s}{\longrightarrow} \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$$

with fibers isomorphic to $\mathbb{P}^1$.

Hence, the object $\Delta^{\widetilde{w}}_{X,\mathsf{R}_0}$ is a cone of a map between constant sheaves on schemes that are smooth over $\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$. Hence, it is ULA.

6.1.6. For $\widetilde{w}$ of length ≥ 2, choose a decomposition

$$\widetilde{w}=\widetilde{w}_1\cdot\widetilde{w}_2, \quad \ell(\widetilde{w})=\ell(\widetilde{w}_1)+\ell(\widetilde{w}_2).$$

Consider the convolution diagram

$$\begin{array}{ccccccc}
 & & & \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \underset{\overrightarrow{h},\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0},\overleftarrow{h}}{\times} \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} & & & \\
 & & \swarrow {\scriptstyle '\overleftarrow{h}} & & {\scriptstyle '\overrightarrow{h}} \searrow & & \\
 & \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} & & & & \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} & \\
 \swarrow {\scriptstyle \overleftarrow{h}\times s} & & {\scriptstyle \overrightarrow{h}\times s} \searrow & & \swarrow {\scriptstyle \overleftarrow{h}\times s} & & {\scriptstyle \overrightarrow{h}\times s} \searrow \\
\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0} & & & \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0} & & & \mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}.
\end{array}$$

By the induction hypothesis, the object

$$\Delta^{\widetilde{w}_1}_{X,\mathsf{R}_0}\widetilde{\boxtimes}\Delta^{\widetilde{w}_2}_{X,\mathsf{R}_0} := ('\overleftarrow{h})^*(\Delta^{\widetilde{w}_1}_{X,\mathsf{R}_0}) \overset{*}{\otimes} ('\overrightarrow{h})^*(\Delta^{\widetilde{w}_2}_{X,\mathsf{R}_0}) \in \mathrm{Shv}(\mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \underset{\overrightarrow{h},\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0},\overleftarrow{h}}{\times} \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0}) \tag{6.1}$$

is ULA with respect to both

$$(\overleftarrow{h}\times s)\circ{}'\overleftarrow{h} \text{ and } (\overrightarrow{h}\times s)\circ{}'\overrightarrow{h}.$$

6.1.7. Consider the convolution map

$$\mathrm{conv}: \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \underset{\overrightarrow{h},\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}, \overleftarrow{h}}{\times} \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \to \mathrm{Hecke}^{\mathrm{Fl}}_{X,\mathsf{R}_0} \underset{\mathrm{Spec}(\mathsf{R}_0)}{\times} X_{\mathsf{R}_0}$$

so that

$$(\overleftarrow{h} \times s) \circ {}'\overleftarrow{h} = (\overleftarrow{h} \times s) \circ \mathrm{conv} \text{ and } (\overrightarrow{h} \times s) \circ {}'\overrightarrow{h} = (\overrightarrow{h} \times s) \circ \mathrm{conv}\,.$$

We obtain that

$$\Delta^{\widetilde{w_1}}_{X,\mathsf{R}_0} \widetilde{\boxtimes} \Delta^{\widetilde{w_2}}_{X,\mathsf{R}_0}$$

is ULA with respect to both

$$(\overleftarrow{h} \times s) \circ \mathrm{conv} \text{ and } (\overrightarrow{h} \times s) \circ \mathrm{conv}\,.$$

Since the map conv is proper, we obtain that

$$\Delta^{\widetilde{w_1}}_{X,\mathsf{R}_0} \star \Delta^{\widetilde{w_2}}_{X,\mathsf{R}_0} := \mathrm{conv}_!(\Delta^{\widetilde{w_1}}_{X,\mathsf{R}_0} \widetilde{\boxtimes} \Delta^{\widetilde{w_2}}_{X,\mathsf{R}_0})$$

is ULA with respect to both $\overleftarrow{h} \times s$ and $\overrightarrow{h} \times s$.

6.1.8. Now, since the map conv induces an isomorphism

$$\mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}_1}_{X,\mathsf{R}_0} \underset{\overrightarrow{h},\mathrm{Bun}^{\mathrm{Fl}}_{G,\mathsf{R}_0}, \overleftarrow{h}}{\times} \mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}_2}_{X,\mathsf{R}_0} \to \mathrm{Hecke}^{\mathrm{Fl},\widetilde{w}}_{X,\mathsf{R}_0},$$

we obtain that

$$\Delta^{\widetilde{w_1}}_{X,\mathsf{R}_0} \star \Delta^{\widetilde{w_2}}_{X,\mathsf{R}_0} \simeq \Delta^{\widetilde{w}}_{X,\mathsf{R}_0}.$$

Hence, $\Delta^{\widetilde{w}}_{X,\mathsf{R}_0}$ is also ULA as required.

□[Theorem 4.2.2]

6.2. **The key mechanism.** We now proceed with the proofs of Theorems 4.4.5 and 5.1.3. The proof will be based on the *contraction principle*, embodied by Proposition 6.2.2 below.

6.2.1. We place ourselves again in the situation of [DG2], over an arbitrary Noetherian base S (i.e., this is a generalization of the context of Sect. 5.2.4, where instead of $\mathrm{Spec}(\mathsf{R}_0)$ we have a more general S).

We claim:

Proposition 6.2.2. *Let* $\mathcal{F} \in \mathrm{Shv}(\mathcal{U})$ *be ULA over* S*. Then so is* $j_!(\mathcal{F}) \in \mathrm{Shv}(\mathcal{Y})$*.*

Proof. Repeats the proof of Proposition 5.2.5.

□

Remark 6.2.3. The assertion of Proposition 6.2.2 replicates that of [HHS, Theorem 6.1.3][17]. We refer the reader to *loc. cit.* where the proof is written out in detail.

Remark 6.2.4. As a side remark, we observe that Proposition 6.2.2 allows us to give an alternative proof of Theorem 4.2.2:

We can show that the objects $\Delta^{\lambda}_{X,\mathsf{R}_0}$ are ULA by reducing to a contractive situation as in the original Kazhdan-Lusztig paper [KL], by intersecting with the opposite Schubert strata.

6.3. **Proof of Theorem 4.4.5.** The proof below is inspired by the computation of IC stalks in [BFGM], and follows the same line of thought as that in [HHS, Theorem 6.2.1].

[17]We are grateful to L. Hamann for pointing this out to us.

6.3.1. Let M denote the Levi quotient of P^-. Let $J \subset I$ be the subset of the Dynkin diagram of G that corresponds to the roots inside M.

Denote by $\Lambda^{\mathrm{pos}}_{G,P}$ the monoid equal to the non-negative integral span of α_i, $i \in (I - J)$. We endow it with the standard order relation.

Recall (see [BG1, Sect. 1.3.3]) that $\widetilde{\mathrm{Bun}}_{P^-}$ carries a stratification indexed by elements of $\Lambda^{\mathrm{pos}}_{G,P}$,

$$\widetilde{\mathrm{Bun}}_{P^-} = \underset{\theta \in \Lambda^{\mathrm{pos}}_{G,P}}{\cup} (\widetilde{\mathrm{Bun}}_{P^-})_\theta,$$

with the open stratum $(\widetilde{\mathrm{Bun}}_{P^-})_0$ being Bun_{P^-}.

For a given θ, let

$$(\widetilde{\mathrm{Bun}}_{P^-})_{<\theta} \subset \widetilde{\mathrm{Bun}}_{P^-} \text{ and } (\widetilde{\mathrm{Bun}}_{P^-})_{\leq\theta} \subset \widetilde{\mathrm{Bun}}_{P^-}$$

be the open substacks

$$\underset{\theta' < \theta}{\cup} (\widetilde{\mathrm{Bun}}_{P^-})_{\theta'} \text{ and } \underset{\theta' \leq \theta}{\cup} (\widetilde{\mathrm{Bun}}_{P^-})_{\theta'},$$

respectively.

We will prove Theorem 4.4.5 by induction on θ. The base of the induction is when $\theta = 0$, in which case $(\widetilde{\mathrm{Bun}}_{P^-})_{\leq\theta} = \mathrm{Bun}_{P^-}$, which is smooth over $\mathrm{Bun}_{M,\mathsf{R}_0}$, and the ULA property[18] is obvious.

Thus, we will assume that the ULA statement holds for

$$j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-,\mathsf{R}_0}})|_{(\widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0})_{<\theta}}$$

and we will deduce its validity for

$$j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-,\mathsf{R}_0}})|_{(\widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0})_{\leq\theta}}.$$

6.3.2. Consider the parabolic Zastava space Zast, see Sect. 9.3.7 below. Pulling back the above stratification along the projection

$$\mathrm{Zast} \to \widetilde{\mathrm{Bun}}_{P^-},$$

we obtain a stratification on Zast. We denote by

$$\mathrm{Zast}_\theta,\ \mathrm{Zast}_{\leq\theta},\ \mathrm{Zast}_{<\theta}$$

the corresponding substacks.

Denote

$$\overset{\circ}{\mathrm{Zast}} := \mathrm{Zast}_0 = \mathrm{Zast} \underset{\widetilde{\mathrm{Bun}}_{P^-}}{\times} \mathrm{Bun}_{P^-}.$$

Denote by j_{Zast} the open embedding $\overset{\circ}{\mathrm{Zast}} \hookrightarrow \mathrm{Zast}$.

Recall also that Zast splits as a disjoint union

$$\mathrm{Zast} := \underset{\theta \in \Lambda^{\mathrm{pos}}_{G,P}}{\sqcup} \mathrm{Zast}^\theta.$$

[18]For the duration of this proof, "ULA" means "ULA over $\mathrm{Bun}_{M,\mathsf{R}_0}$".

6.3.3. Arguing as in [BFGM, Sect. 5] or [Ga4, Sect. 3.9], we obtain:

Lemma 6.3.4.
(a) *The ULA property of* $j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-,\mathsf{R}_0}})|_{(\widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0})_{<\theta}}$ *implies the ULA property of*

$$(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}_{\mathsf{R}_0})_{<\theta}}.$$

(b) *The following statements are equivalent:*

(i) $j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-,\mathsf{R}_0}})|_{(\widetilde{\mathrm{Bun}}_{P^-,\mathsf{R}_0})_{\leq\theta}}$ *is ULA;*

(ii) $(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}_{\mathsf{R}_0})_{\leq\theta}}$ *is ULA;*

(iii) $(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}^\theta_{\mathsf{R}_0})_{\leq\theta}}$ *is ULA.*

6.3.5. Let us denote by $j_{<\theta,\leq\theta}$ the open embedding

$$\mathrm{Zast}^\theta_{<\theta} \hookrightarrow \mathrm{Zast}^\theta_{\leq\theta}.$$

Note that the complement of this embedding is the closed stratum $\mathrm{Zast}^\theta_\theta \subset \mathrm{Zast}^\theta_{\leq\theta}$.

Thus, we have to show

$$(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}^\theta_{\mathsf{R}_0})_{<\theta}} \text{ is ULA } \Rightarrow (j_{<\theta,\leq\theta})_!\left((j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}^\theta_{\mathsf{R}_0})_{<\theta}}\right) \text{ is ULA }. \tag{6.2}$$

6.3.6. Recall now (see [BFGM, Sect. 5.1]) that $\mathrm{Zast}^\theta = \mathrm{Zast}^\theta_{\leq\theta}$ carries an action of $\mathbb{G}_m$ that contracts it onto the locus $\mathrm{Zast}^\theta_\theta$.

Since the object $(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}})|_{(\mathrm{Zast}^\theta_{\mathsf{R}_0})_{<\theta}}$ is $\mathbb{G}_m$-equivariant, the implication (6.2) follows from Proposition 6.2.2.

□[Theorem 4.4.5]

6.4. **Proof of Theorem 5.1.3.** It is easy to reduce the assertion to the case when G is semi-simple, which we will now assume.[19]

6.4.1. Recall that according to [Sch], the diagonal map

$$\Delta : \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$$

can be factored as

$$\mathrm{Bun}_G = \mathrm{Bun}_G \times \mathrm{pt} \to \mathrm{Bun}_G \times \mathrm{pt}/Z_G \overset{j}{\hookrightarrow} \overline{\mathrm{Bun}}_G \overset{\overline{\Delta}}{\to} \mathrm{Bun}_G \times \mathrm{Bun}_G,$$

where the map j is an open embedding and $\overline{\Delta}$ is proper.

6.4.2. *Example.* For $G = SL_2$, the stack $\overline{\mathrm{Bun}}_G$ classifies triples

$$(\mathcal{E}_1, \mathcal{E}_2, \alpha)/\mathbb{G}_m,$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ rank-2 bundles with trivialized determinants, and α is a *non-zero* map $\mathcal{E}_1 \to \mathcal{E}_2$. The action of $\mathbb{G}_m$ is given by scaling α.

The open locus $\mathrm{Bun}_G \times \mathrm{pt}/Z_G$ corresponds to the condition that α be an isomorphism.

6.4.3. Thus, it suffices to show that the object

$$j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}} \boxtimes R_{Z_G}) \in \mathrm{Shv}(\overline{\mathrm{Bun}}_{G,\mathsf{R}_0}) \tag{6.3}$$

is ULA over $\mathrm{Spec}(\mathsf{R}_0)$, where

$$R_{Z_G} \in \mathrm{Rep}(Z_G) \to \mathrm{Shv}(\mathrm{Spec}(\mathsf{R}_0)/Z_G)$$

denotes the regular representation.

[19] Otherwise replace R_{Z_G} below by the !-direct image of e along the map $\mathrm{pt} \to \mathrm{pt}/Z_G$.

6.4.4. Recall (see [Sch, Sect. 2.2.7]) that stack $\overline{\mathrm{Bun}}_G$ admits a stratification indexed by the poset of standard parabolics in G:

$$\overline{\mathrm{Bun}}_G = \underset{P}{\cup} \overline{\mathrm{Bun}}_{G,P}$$

with the open stratum $\overline{\mathrm{Bun}}_{G,G}$ being the image of the embedding j.

Denote by $\overline{\mathrm{Bun}}_{G,\geq P}$ (resp., $\overline{\mathrm{Bun}}_{G,>P}$) the open substack equal to the union of the strata corresponding to the parabolics P' with $P \subset P'$ (resp., $P \subsetneq P'$).

We argue by induction and assume that the object (6.3) is ULA when restricted to the open substack $\overline{\mathrm{Bun}}_{G,>P,\mathsf{R}_0}$. We now perform the induction step and prove that the ULA property holds over $\overline{\mathrm{Bun}}_{G,\geq P,\mathsf{R}_0}$.

6.4.5. Consider the stratum $\overline{\mathrm{Bun}}_{G,P}$. According to [Sch, Sect. 3] that $\overline{\mathrm{Bun}}_{G,P}$ admits a further stratification indexed by elements of $\Lambda^{\mathrm{pos}}_{G,P}$:

$$\overline{\mathrm{Bun}}_{G,P} = \underset{\theta\in\Lambda^{\mathrm{pos}}_{G,P}}{\cup} (\overline{\mathrm{Bun}}_{G,P})_\theta.$$

For a given θ, let

$$(\overline{\mathrm{Bun}}_{G,P})_{<\theta} \subset (\overline{\mathrm{Bun}}_{G,P})_{\leq\theta}$$

be the corresponding open subsets.

In particular, we have the open locus

$$(\overline{\mathrm{Bun}}_{G,P})_0 \subset \overline{\mathrm{Bun}}_{G,P}.$$

6.4.6. *Example.* In the example of $G = SL_2$, the Borel stratum corresponds to the condition that the map α has generic rank 1 and hence factors as

$$\mathcal{E}_1 \overset{\beta_1}{\twoheadrightarrow} \mathcal{L}_1 \overset{\gamma}{\to} \mathcal{L}_2 \overset{\beta_2}{\hookrightarrow} \mathcal{E}_2,$$

where $\mathcal{L}_i$ are line bundles and β_i are bundle maps.

The stratification by $\Lambda^{\mathrm{pos}}_{G,P} = \mathbb{Z}^{\geq 0}$ is given by the total degree of zeroes of the map γ.

6.4.7. Denote by $\overline{\mathrm{Bun}}_{G,\geq P,<\theta}$ and $\overline{\mathrm{Bun}}_{G,\geq P,\leq\theta}$ the open substacks

$$\overline{\mathrm{Bun}}_{G,>P} \cup (\overline{\mathrm{Bun}}_{G,P})_{<\theta} \text{ and } \overline{\mathrm{Bun}}_{G,>P} \cup (\overline{\mathrm{Bun}}_{G,P})_{\leq\theta},$$

respectively.

By induction, we can assume that (6.3) is ULA[20] when restricted to $\overline{\mathrm{Bun}}_{G,\geq P,<\theta,\mathsf{R}_0}$. We will now perform the induction step and prove that the ULA property holds over $\overline{\mathrm{Bun}}_{G,\geq P,\leq\theta,\mathsf{R}_0}$.

Remark 6.4.8. The rest of the argument, which is explained below, uses the same principle as the proof of Theorem 4.4.5: we will replace $\overline{\mathrm{Bun}}_{G,\geq P,\mathsf{R}_0}$ by its local model and reduce the assertion to a contractive situation (i.e., one covered by Proposition 6.2.2).

6.5. **Proof of Theorem 5.1.3, continuation.**

6.5.1. Let Y^P denote the open substack

$$\left(\mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}\right)^{\mathrm{tr}} \subset \mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-},$$

where the superscript "tr" refers to the generic transversality condition, see [Sch, Sect. 6.1.6].

[20]For the duration of this proof, "ULA" means "ULA over $\mathrm{Spec}(\mathsf{R}_0)$.

6.5.2. *Example.* For $G = SL_2$, the fiber product

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}$$

classifies the data of

$$\mathcal{L}'_1 \overset{\delta_1}{\to} \mathcal{E}_1 \overset{\alpha}{\to} \mathcal{E}_2 \overset{\delta_2}{\to} \mathcal{L}'_2,$$

where δ_i are bundle maps.

The generic transversality condition is that the composite map $\mathcal{L}'_1 \to \mathcal{L}'_2$ be non-zero.

6.5.3. The stratifications on $\overline{\mathrm{Bun}}_G$ induce the corresponding stratifications on Y^P, to be denoted $Y^P_{P'}$, $Y^P_{P',\theta}$, etc. We note that $Y^P_{P'} = \emptyset$ unless $P \subset P'$.

Note that the open stratum Y^P_G identifies with

$$\overset{\circ}{\mathrm{Zast}} \times \mathrm{pt}/Z_G.$$

Denote by j_Y the open embedding

$$\overset{\circ}{\mathrm{Zast}} \times \mathrm{pt}/Z_G \hookrightarrow Y^P.$$

In addition, Y^P splits as a disjoint union

$$Y^P := \bigsqcup_{\theta \in \Lambda^{\mathrm{pos}}_{G,P}} Y^{P,\theta}, \tag{6.4}$$

and $Y^{P,\theta}_{P,\theta'}$ is empty unless $\theta' \leq \theta$.

6.5.4. *Example.* In the example of $G = SL_2$, the decomposition (6.4) is according to the total degree of zeroes of the map $\mathcal{L}'_1 \to \mathcal{L}'_2$.

6.5.5. The following is parallel to Lemma 6.3.4:

Lemma 6.5.6.

(a) *The ULA property of*

$$j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}} \boxtimes R_{Z_G})|_{\overline{\mathrm{Bun}}_{G,\geq P,<\theta,\mathsf{R}_0}}$$

implies the ULA property of

$$(j_Y)_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^P_{\geq P,<\theta,\mathsf{R}_0}}.$$

(b) *The following conditions are equivalent:*

(i) $j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{G,\mathsf{R}_0}} \boxtimes R_{Z_G})|_{\overline{\mathrm{Bun}}_{G,\geq P,\leq\theta,\mathsf{R}_0}}$ *is ULA;*

(ii) $(j_Y)_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^P_{\geq P,\leq\theta,\mathsf{R}_0}}$ *is ULA;*

(iii) $(j_Y)_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^{P,\theta}_{\geq P,\leq\theta,\mathsf{R}_0}}$ *is ULA.*

6.5.7. Let $j_{\geq P,<\theta,\leq\theta}$ the open embedding

$$Y^{P,\theta}_{\geq P,<\theta} \hookrightarrow Y^{P,\theta}_{\geq P,\leq\theta}.$$

Note that the complement of this embedding is the closed stratum

$$Y^{P,\theta}_{P,\theta} \subset Y^{P,\theta}_{\geq P,\leq\theta}.$$

6.5.8. *Example.* In the example of $G = SL_2$, the closed substack $Y^{P,\theta}_{P,\theta}$ is the locus where the maps

$$\mathcal{L}'_1 \to \mathcal{L}_1 \text{ and } \mathcal{L}_2 \to \mathcal{L}'_2$$

are isomorphisms.

So $\mathcal{E}_i = \mathcal{L}_i \oplus \mathcal{L}_i^{\otimes -1}$, and the map α is

$$\mathcal{E}_1 \simeq \mathcal{L}_1 \oplus \mathcal{L}_1^{\otimes -1} \twoheadrightarrow \mathcal{L}_1 \to \mathcal{L}_2 \hookrightarrow \mathcal{L}_2 \oplus \mathcal{L}_2^{\otimes -1}.$$

Thus, we obtain that $Y^{P,\theta}_{P,\theta}$ is isomorphic to a version of the Hecke stack for $\mathbb{G}_m$: it classifies pairs $(\mathcal{L}_1, D)$, where $\mathcal{L}_1$ is a line bundle and D is an effective divisor on X of degree θ.

6.5.9. Thus, we have to show:

$$(j_Y)_!(\mathsf{e}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^{P,\theta}_{\geq P,<\theta,\mathsf{R}_0}} \text{ is ULA } \Rightarrow (j_{\geq P,<\theta,\leq\theta})_!\left((j_Y)_!(\mathsf{e}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^{P,\theta}_{\geq P,<\theta,\mathsf{R}_0}}\right) \text{ is ULA.}$$

6.5.10. Recall now (see [Sch, Sect. 6.5.5]) that $Y^{P,\theta}_{\geq P,\leq\theta}$ carries an action of $\mathbb{G}_m$, which contracts its into $Y^{P,\theta}_{P,\theta}$.

6.5.11. *Example.* In the example of $G = SL_2$, the above action is the following one. A scalar $c \in \mathbb{G}_m$ acts on the triple

$$\mathcal{L}'_1 \overset{\delta_1}{\to} \mathcal{E}_1 \overset{\alpha}{\to} \mathcal{E}_2 \overset{\delta_2}{\to} \mathcal{L}'_2$$

by

$$\delta_1 \mapsto c^{-1} \cdot \delta_1,\ \delta_2 \mapsto c^{-1} \cdot \delta_2,\ \alpha \mapsto c^2 \cdot \alpha.$$

6.5.12. Since the object

$$(j_Y)_!(\mathsf{e}_{\overset{\circ}{\mathrm{Zast}}_{\mathsf{R}_0}} \boxtimes R_{Z_G})|_{Y^P_{\geq P,<\theta,\mathsf{R}_0}} \in \mathrm{Shv}(Y^P_{\geq P,<\theta,\mathsf{R}_0})$$

is *quasi-equivariant* with respect to this $\mathbb{G}_m$-action (i.e., equivariant after passing to a finite self-isogeny of $\mathbb{G}_m$), the required assertion follows from Proposition 6.2.2.

$\square$[Theorem 5.1.3]

7. Proof of Theorem 4.4.2

We will give two proofs. The first proof is shorter, but it requires more stringent assumptions on the characteristic of the ground field k (see Assumption (*) in Sect. 7.1.3 and Remark 7.1.5).

7.1. **First proof of Theorem 4.4.2.**

7.1.1. Let

$$\exp^{\boxtimes I}/T \in \mathrm{Shv}(\mathbb{G}_a^I/T)$$

be as in [GLC1, Sect. 3.3], where I is the Dynkin diagram of G (see also Sect. 8.4.11 below).

Recall that we have a canonically defined T-equivariant map

$$\chi^I : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathbb{G}_a^I,$$

and consider the corresponding map

$$\chi^I/T : \mathrm{Bun}_{N,\rho(\omega_X)}/T \to \mathbb{G}_a^I/T.$$

Denote

$$\exp_{\chi^I}/T := (\chi^I/T)^*(\exp^{\boxtimes I}/T) \in \mathrm{Shv}(\mathrm{Bun}_{N,\rho(\omega_X)}/T).$$

Recall the map

$$\mathsf{p} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G$$

and note that it naturally factors via a map

$$\mathsf{p}/T : \mathrm{Bun}_{N,\rho(\omega_X)}/T \to \mathrm{Bun}_G.$$

Recall that the object $\mathrm{Poinc}_!^{\mathrm{Vac}}$ is defined as

$$(\mathsf{p}/T)_!(\exp_{\chi^I}/T) \in \mathrm{Shv}(\mathrm{Bun}_G).$$

7.1.2. We consider the above objects over $\mathrm{Spec}(\mathsf{R}_0)$. The object $\exp^{\boxtimes I}_{\mathsf{R}_0}/T$ is ULA over $\mathrm{Spec}(\mathsf{R}_0)$ (e.g., by Proposition 6.2.2). Since the map χ^I/T is smooth, we obtain that

$$\exp_{\chi^I,\mathsf{R}_0}/T \in \mathrm{Shv}(\mathrm{Bun}_{N,\rho(\omega_X),\mathsf{R}_0}/T)$$

is also ULA.

For the rest of the first proof, we will consider separately the cases of $g \geq 2$, $g = 1$ and $g = 0$.

7.1.3. We will make the following assumption on the pair $(G, \mathrm{char}(p))$:

(*) *For every standard Levi M and a central character $\check{\mu}$ of Z_M, the direct summand $(\mathfrak{n}_P)_{\check{\mu}}$, viewed as a representation of M, has* strangeness 0 *(see* [DG1, Definition 10.3.4] *for what this means).*

Remark 7.1.4. As was explained in *loc. cit.*, the above assumption is automatic when one works over a ground field of characteristic 0.

Remark 7.1.5. According to [IMP, Theorem 3.1], the following condition guarantees that Assumption (*) is satisfied:

We need that for all M and all roots $\check{\alpha}$ appearing in $\mathfrak{n}_P$,

$$2 \cdot \langle \rho_M, \check{\alpha} \rangle \cdot |Z^0_M \cap [M, M]| < p.$$

In the above formula, the factor $|Z^0_M \cap [M, M]|$ appears since in [IMP] only representations with trivial determinant are allowed.

7.1.6. We proceed with the first proof of Theorem 4.4.2. We start with the case $g \geq 2$.

Note that the map p/T factors as

$$\mathrm{Bun}_{N,\rho(\omega_X)}/T \to \mathrm{Bun}_G^{(\leq (2g-2)\cdot\rho)} \overset{\jmath_{(2g-2)\cdot\rho}}{\hookrightarrow} \mathrm{Bun}_G,$$

where the first arrow is proper (see [DG1, Theorem 7.4.3(a)]).

Hence, it suffices to show that the functor

$$(\jmath_{(2g-2)\cdot\rho})_! : \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{R}_0}^{(\leq (2g-2)\cdot\rho)}) \to \mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{R}_0})$$

preserves the property of being ULA over $\mathrm{Spec}(\mathsf{R}_0)$.

We now use Assumption (*). It implies that the complement of the embedding

$$\mathrm{Bun}_G^{(\leq (2g-2)\cdot\rho)} \overset{\jmath_{(2g-2)\cdot\rho}}{\hookrightarrow} \mathrm{Bun}_G$$

is contractive, see [DG1, Proposition 10.1.3].

Hence, the required preservation of the ULA property follows from Proposition 6.2.2.

7.1.7. We now consider the case $g = 1$. In this case, the map p/T factors as

$$\mathrm{Bun}_{N,\rho(\omega_X)}/T \to \mathrm{Bun}_G^{(\leq 0)} \overset{\jmath_0}{\hookrightarrow} \mathrm{Bun}_G. \tag{7.1}$$

Note that $\mathrm{Bun}_G^{(\leq 0)}$ is the semi-stable locus $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$. We factor the first arrow in (7.1) as

$$\mathrm{Bun}_{N,\rho(\omega_X)}/T \overset{\omega \text{ is trivial}}{\simeq} \mathrm{Bun}_N/T \hookrightarrow \mathrm{Bun}_B^0 \to \mathrm{Bun}_G^{ss},$$

where the second arrow is a closed embedding, and Bun_B^0 is the preimage of the neutral connected component of Bun_T under $\mathsf{q} : \mathrm{Bun}_B \to \mathrm{Bun}_T$.

Note that the map

$$\mathrm{Bun}_B^0 \to \mathrm{Bun}_G^{ss}$$

is proper (indeed, $\mathrm{Bun}_B^0 = \mathrm{Bun}_B^0 \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{ss} \to \overline{\mathrm{Bun}}_B^0 \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{ss}$ is an equality).

Hence, the first arrow in (7.1) is proper. Therefore, it remains to show that the functor

$$(\jmath_0)_! : \text{Shv}(\text{Bun}_{G,\mathsf{R}_0}^{(\leq 0)}) \to \text{Shv}(\text{Bun}_{G,\mathsf{R}_0})$$

preserves the property of being ULA over $\text{Spec}(\mathsf{R}_0)$.

However, in genus 1, the complement of $\text{Bun}_G^{(\leq 0)}$ in Bun_G is contractive (by Assumption (*) and [DG1, Proposition 10.1.3]). Hence, the assertion follows from Proposition 6.2.2.

7.2. **Proof of Proposition 4.4.2 in genus** 0**.** It remains to consider the case of $g = 0$. Here the argument will be of different nature and in fact can be considered as a simplified version of the second proof relying on special features of the genus 0 situation.

7.2.1. We will show directly that

$$\Phi(\text{Poinc}_{!,\mathsf{R}_0}^{\text{Vac}}) = 0$$

(which is what we need for Property (C)).

However, by Remark 4.1.10, this is equivalent to the fact that $\text{Poinc}_{!,\mathsf{R}_0}^{\text{Vac}}$ is ULA over $\text{Spec}(\mathsf{R}_0)$.

7.2.2. Note that the direct sum of the constant terms functors

$$\text{CT}_*^\lambda : \text{Shv}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_T^\lambda), \quad \lambda \in \Lambda^+$$

is conservative when $g = 0$.

Hence, it suffices to show that

$$\text{CT}_*^\lambda \circ \Phi(\text{Poinc}_{!,\mathsf{R}_0}^{\text{Vac}}) = 0, \quad \lambda \in \Lambda^+.$$

7.2.3. Recall also that according to [DG2], we have a canonical isomorphism

$$\text{CT}_*^\lambda \simeq \text{CT}_!^{-,\lambda}. \tag{7.2}$$

We claim now that the canonical maps

$$\text{CT}_!^{-,\lambda} \circ \Phi \to \Phi \circ \text{CT}_!^{-,\lambda}$$

are isomorphisms when $\lambda \in \Lambda^+$.

Indeed, the functor $\text{CT}_!^{-,\lambda}$ is given by

$$(\mathsf{q}^-)_! \circ (\mathsf{p}^-)^*$$

for the diagram

$$\text{Bun}_G \overset{\mathsf{p}^-}{\leftarrow} \text{Bun}_{B^-}^\lambda \overset{\mathsf{q}^-}{\to} \text{Bun}_T^\lambda.$$

Now, for $\lambda \in \Lambda^+$, the map p^- is smooth, and hence the functor $(\mathsf{p}^-)^*$ commutes with vanishing cycles. The functor $(\mathsf{q}^-)_!$ commutes with vanishing cycles thanks to the contraction principle (see [DG2, Sect. 4.1]) and isomorphism (5.2).

7.2.4. Thus, it suffices to show that

$$\Phi \circ \text{CT}_!^{-,\lambda}(\text{Poinc}_{!,\mathsf{R}_0}^{\text{Vac}}) = 0.$$

However, this is the assertion of (7.6) below.

□[Theorem 4.4.2]

7.3. **Second proof of Theorem 4.4.2.** It is easy to reduce the assertion to the case when G is semi-simple, so we will make this assumption.

7.3.1. Recall (see Remark 4.1.10) that we have to show that

$$\Phi(\text{Poinc}_{!,\mathsf{R}_0}^{\text{Vac}}) = 0. \tag{7.3}$$

7.3.2. Recall that we have the semi-orthogonal decomposition of the category $\mathrm{Shv}(\mathrm{Bun}_G)$

$$\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{Eis}} \overset{\mathbf{e}_{\mathrm{Eis}}}{\hookrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathbf{e}_{\mathrm{cusp}}}{\hookleftarrow} \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{cusp}}.$$

Thus, every object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ fits into a (canonically defined) fiber sequence

$$\mathbf{e}_{\mathrm{Eis}} \circ \mathbf{e}^R_{\mathrm{Eis}}(\mathcal{F}) \to \mathcal{F} \to \mathbf{e}_{\mathrm{cusp}} \circ \mathbf{e}^L_{\mathrm{cusp}}(\mathcal{F}).$$

Furthermore,

$$\mathbf{e}^R_{\mathrm{Eis}}(\mathcal{F}) = 0 \text{ if and only if } \mathrm{CT}_*(\mathcal{F}) = 0 \text{ for all proper parabolics } P \subset G.$$

7.3.3. We will show that

$$\mathbf{e}^L_{\mathrm{cusp}}(\Phi(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0})) = 0 \tag{7.4}$$

and

$$\mathrm{CT}_*(\Phi(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0})) = 0. \tag{7.5}$$

By the above, this will imply (7.3).

7.4. **The Eisenstein part.** We first tackle (7.5).

7.4.1. First, we claim:

Proposition 7.4.2. *The natural transformation*

$$\Psi \circ \mathrm{CT}_* \to \mathrm{CT}_* \circ \Psi$$

is an isomorphism.

Proof. Using Beilinson's definition of nearby cycles, the functor Ψ is the colimit of functors

$$\mathbf{i}_0^* \circ (\mathbf{j}_0)_*((-) \otimes \mathcal{E}),$$

where:

- The maps $\mathbf{i}_0$ and $\mathbf{j}_0$ are $\mathrm{Spec}(\mathsf{k}) \to \mathrm{Spec}(\mathsf{R}_0)$ and $\mathrm{Spec}(\mathsf{K}_0) \to \mathrm{Spec}(\mathsf{R}_0)$ and base changes thereof;
- $\mathcal{E}$ is a local system on $\mathrm{Spec}(\mathsf{K}_0)$ or a pullback thereof.

We claim that the operation CT_* commutes with all the three functors involved:

(i) The commutation with $(-) \otimes \mathcal{E}$ is obvious;

(ii) The commutation with the functor $(\mathbf{j}_0)_*$ is also obvious, since CT_* involves !-pullbacks and *-pushforwards.

(iii) In order to establish the commutation with $\mathbf{i}_0^*$, we apply the isomorphism (7.2), and rewrite CT_* as $\mathrm{CT}^-_!$. Now again the commutation becomes obvious, since $\mathrm{CT}^-_!$ involves *-pullback and !-pushforward. □

Remark 7.4.3. Note that a variation of this argument shows that the natural transformation

$$\mathrm{Sp} \circ \mathrm{CT}_* \to \mathrm{CT}_* \circ \mathrm{Sp}$$

is an isomorphism.

Corollary 7.4.4. *The natural transformation*

$$\Phi \circ \mathrm{CT}_* \to \mathrm{CT}_* \circ \Phi$$

is an isomorphism.

Proof. Follows from the fiber sequence

$$\Phi \to \Psi \to \mathbf{i}^!,$$

where the functor $\mathbf{i}^!$ obviously commutes with CT_*.

□

7.4.5. By Corollary 7.4.4, it suffices to show that

$$\Phi \circ \mathrm{CT}_*(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0})) = 0.$$

We use again the isomorphism (7.2), and we rewrite

$$\mathrm{CT}_*(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) \simeq \mathrm{CT}^-_!(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}).$$

Thus, we have to prove:

$$\Phi \circ \mathrm{CT}^-_!(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) = 0. \tag{7.6}$$

We claim:

Theorem 7.4.6. *There is a canonical isomorphism*

$$(\mathrm{transl}_{\rho_P(\omega_X)})^* \circ \mathrm{CT}^-_!(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!})[d] \simeq \mathrm{Fact}(\Omega^{\mathrm{loc}}) \star \mathrm{Poinc}^{\mathrm{Vac}}_{M,!},$$

where:

- $\mathrm{transl}_{\rho_P(\omega_X)}$ *is the automorphism of* Bun_M *given by translation by* $\rho_P(\omega_X) \in \mathrm{Bun}_{Z_M}$*;*
- *The integer* d *is as in* [GLC3, Corollary 10.1.8]*;*
- *The functor* $(\mathrm{Fact}(\Omega^{\mathrm{loc}}) \star (-))$ *is as in Theorem 9.2.7.*

The assertion of Theorem 7.4.6 is obtained from that of Theorem 9.2.7 proven below by a duality manipulation (see [GLC3, Corollary 10.1.8]).

Remark 7.4.7. Alternatively, one can prove Theorem 7.4.6 by rerunning the argument of Theorem 9.2.7. However, the proof is simpler as here one works with the open Zastava space $\overset{\circ}{\mathrm{Z}}\mathrm{ast}$ and one does not need any local acyclicity assertions.

7.4.8. Thus, we have to show that

$$\Phi(\mathrm{Fact}(\Omega^{\mathrm{loc}}) \star \mathrm{Poinc}^{\mathrm{Vac}}_{M,!,\mathsf{R}_0}) = 0.$$

By Proposition 4.2.5,

$$\Phi(\mathrm{Fact}(\Omega^{\mathrm{loc}}) \star \mathrm{Poinc}^{\mathrm{Vac}}_{M,!,\mathsf{R}_0}) \simeq \mathrm{Fact}(\Omega^{\mathrm{loc}}) \star \Phi(\mathrm{Poinc}^{\mathrm{Vac}}_{M,!,\mathsf{R}_0}).$$

Now, by induction on the semi-simple rank, we can assume that $\Phi(\mathrm{Poinc}^{\mathrm{Vac}}_{M,!,\mathsf{R}_0}) = 0$, and the assertion follows.

7.5. **The cuspidal part.**

7.5.1. The statement that we want to prove is that the map

$$\mathbf{e}^L_{\mathrm{cusp}}(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{k}}) \simeq \mathbf{e}^L_{\mathrm{cusp}} \circ \mathbf{i}^*_0(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) \to \mathbf{e}^L_{\mathrm{cusp}} \circ \Psi(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{K}_0}) \tag{7.7}$$

is an isomorphism.

Thanks to Theorem 4.4.5, the functors $\mathbf{i}^*_0$, $\mathbf{j}_{0,*}$, and Ψ are defined on the cuspidal category, viewed as a *quotient* of $\mathrm{Shv}(\mathrm{Bun}_G)$, i.e., in a way that commutes with the projection

$$\mathbf{e}^L_{\mathrm{cusp}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{cusp}}.$$

7.5.2. Recall that $\mathrm{Poinc}^{\mathrm{Vac}}_! \in \mathrm{Shv}(\mathrm{Bun}_G)$ was defined as

$$(\mathsf{p}/T)_!(\exp_{\chi^I}/T) := (\chi^I/T)^*(\exp^{\boxtimes I}/T),$$

where $\exp^{\boxtimes I}/T$ is as in [GLC1, Sect. 3.3].

We will now change the notations

$$\exp^{\boxtimes I}/T \rightsquigarrow \exp^{\boxtimes I}_!/T \text{ and } \exp_{\chi^I}/T \rightsquigarrow \exp_{\chi^I,!}/T$$

and we note that there exists another object

$$\exp^{\boxtimes I}_*/T \in \mathrm{Shv}(\mathbb{G}_a^I/T),$$

see Sect. 8.4.11.

In addition, there is a canonically defined map

$$\exp^{\boxtimes I}_!/T \to \exp^{\boxtimes I}_*/T, \tag{7.8}$$

see (8.8).

7.5.3. Denote

$$\exp_{\chi^I,*}/T := (\chi^I/T)^*(\exp^{\boxtimes I}_*/T).$$

Let

$$\mathrm{Poinc}^{\mathrm{Vac}}_* \in \mathrm{Shv}(\mathrm{Bun}_G)$$

be the object equal to

$$(\mathsf{p}/T)_*(\exp_{\chi^I,*}/T).$$

The map (7.8) gives rise to a map

$$\mathrm{Poinc}^{\mathrm{Vac}}_! \to \mathrm{Poinc}^{\mathrm{Vac}}_*. \tag{7.9}$$

We will prove:

Theorem 7.5.4. *The cone of* (7.9) *belongs to* $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{Eis}}$.

Actually, the proof shows more, namely that for any $\mathcal{G} \in \mathrm{Shv}(\mathrm{Spec}(\mathsf{R}_0))$, the morphism

$$\mathsf{p}_{\mathsf{R}_0,!}(\pi^*_{\mathrm{Bun}_{N,\rho(\omega),\mathsf{R}_0}}(\mathcal{G}) \overset{*}{\otimes} \exp_{\chi^I,!}/T) \to \mathsf{p}_{\mathsf{R}_0,*}(\pi^*_{\mathrm{Bun}_{N,\rho(\omega),\mathsf{R}_0}}(\mathcal{G}) \overset{*}{\otimes} \exp_{\chi^I,*}/T) \tag{7.10}$$

has cone lying in $\mathrm{Shv}(\mathrm{Bun}_{G,\mathsf{R}_0})_{\mathrm{Eis}}$; here $\pi_{\mathrm{Bun}_{N,\rho(\omega),\mathsf{R}_0}}$ is the projection from $\mathrm{Bun}_{N,\rho(\omega),\mathsf{R}_0}$ to $\mathrm{Spec}(\mathsf{R}_0)$.

The proof of Theorem 7.5.4 will be given in Sect. 8. We now proceed with the proof of the fact that (7.7) is an isomorphism.

7.5.5. Take $\mathcal{E}$ to be a lisse sheaf on $\mathrm{Spec}(\mathsf{K}_0)$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0} \overset{*}{\otimes} \mathbf{j}_{0,*}(\mathcal{E}) & \longrightarrow & \mathbf{j}_{0,*}(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{K}_0} \overset{*}{\otimes} \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Poinc}^{\mathrm{Vac}}_{*,\mathsf{R}_0} \overset{*}{\otimes} \mathbf{j}_{0,*}(\mathcal{E}) & \longrightarrow & \mathbf{j}_{0,*}(\mathrm{Poinc}^{\mathrm{Vac}}_{*,\mathsf{K}_0} \overset{*}{\otimes} \mathcal{E}). \end{array} \tag{7.11}$$

The vertical arrows have Eisenstein cones by Theorem 7.5.4. Moreover, the composition from upper left to bottom right has Eisenstein cone by the property of (7.10) stated above. Therefore, each arrow in the above square becomes an isomorphism after applying $\mathbf{e}^L_{\mathrm{cusp}}$.

Applying $\mathbf{i}_0^*$, we obtain a similar diagram

$$\begin{array}{ccc} \mathbf{i}_0^*(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) \overset{*}{\otimes} \mathbf{i}_0^* \circ \mathbf{j}_{0,*}(\mathcal{E}) & \longrightarrow & \mathbf{i}_0^* \circ \mathbf{j}_{0,*}(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{K}_0} \overset{*}{\otimes} \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathbf{i}_0^*(\mathrm{Poinc}^{\mathrm{Vac}}_{*,\mathsf{R}_0}) \overset{*}{\otimes} \mathbf{i}_0^* \circ \mathbf{j}_{0,*}(\mathcal{E}) & \longrightarrow & \mathbf{i}_0^* \circ \mathbf{j}_{0,*}(\mathrm{Poinc}^{\mathrm{Vac}}_{*,\mathsf{K}_0} \overset{*}{\otimes} \mathcal{E}) \end{array} \tag{7.12}$$

in which again all arrows have Eisenstein cones. By Beilinson's construction of nearby cycles, we can pass to a colimit of such $\mathcal{E}$'s so that the top horizontal arrow in the above diagram becomes the map

$$\mathbf{i}_0^*(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) = \mathbf{i}_0^*(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{R}_0}) \otimes \Psi(\mathsf{e}_{\mathrm{Spec}(K_0)}) \to \Psi(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathsf{K}_0}),$$

which we deduce has Eisenstein cone.

8. Comparison of !- vs *- Poincaré objects

The goal of this section is to prove Theorem 7.5.4. We continue to assume that G is semi-simple.

8.1. **The case when there exists the exponential sheaf.** Note that we only used Theorem 7.5.4 over K and R. However, we will first give a proof over a field of positive characteristic (or for D-modules), since it conveys the intuitive picture.

Remark 8.1.1. In the course of the proof, we will see that the cone of (7.9) admits a canonical filtration, whose associated graded can be described explicitly, see Remark 8.2.8.

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8.1.2. We start by rewriting the objects

$$\exp_!^{\boxtimes I}/T \text{ and } \exp_*^{\boxtimes I}/T \tag{8.1}$$

in terms of the exponential sheaf[21].

Namely, we start with

$$\exp^{\boxtimes I} \in \mathrm{Shv}(\mathbb{G}_a^I)$$

(here I is the set of vertices of the Dynkin diagram) and the objects (8.1) are its !- and *- direct images, respectively, along the map

$$\mathbb{G}_a^I \to \mathbb{G}_a^I/T.$$

8.1.3. Consider the stack

$$\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I.$$

Let f denote the projection

$$\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I \to \overline{\mathrm{Bun}}_{N,\rho(\omega)}.$$

We consider the canonical T-action on $\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I$, where the action on the second factor is via

$$T \to T_{\mathrm{adj}} \overset{\text{simple roots}}{\simeq} \mathbb{G}_m^I.$$

Let f/T denote the projection

$$(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)/T \to \overline{\mathrm{Bun}}_{N,\rho(\omega)}/T.$$

8.1.4. Let $\mathbf{j}$ denote the open embedding

$$\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^I \hookrightarrow \overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I.$$

and $\mathbf{j}/T$ the embedding

$$\overline{\mathrm{Bun}}_{N,\rho(\omega)}/Z_G \simeq (\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^I)/T \hookrightarrow (\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)/T.$$

We can rewrite

$$\mathrm{Poinc}^{\mathrm{Vac}}_? \simeq (\overline{\mathsf{p}}/T)_? \circ (f/T)_? \circ (\mathbf{j}/T)_? \circ (\pi_{Z_G})_? \circ j_? \circ (\chi^I)^*(\exp^{\boxtimes \mathrm{I}}), \tag{8.2}$$

where:

- ? is either ! or $*$;
- j denote the embedding $\mathrm{Bun}_{N,\rho(\omega)} \hookrightarrow \overline{\mathrm{Bun}}_{N,\rho(\omega)}$;
- π_{Z_G} denotes the projection $\overline{\mathrm{Bun}}_{N,\rho(\omega)} \to \overline{\mathrm{Bun}}_{N,\rho(\omega)}/Z_G$.

[21]The definition of these objects in Sect. 8.4.11 uses the Kirillov model and hence avoids the exponential sheaf.

Note that:

- The map $\overline{\mathsf{p}}/T$ is proper, so $(\overline{\mathsf{p}}/T)_! \to (\overline{\mathsf{p}}/T)_*$ is an isomorphism;
- The map π_{Z_G} is finite[22], so $(\pi_{Z_G})_! \to (\pi_{Z_G})_*$ is an isomorphism;
- The extension of $(\chi^I)^*(\exp^{\boxtimes I})$ along j is clean, so the map
$$j_! \circ (\chi^I)^*(\exp^{\boxtimes I}) \to j_* \circ (\chi^I)^*(\exp^{\boxtimes I})$$
is an isomorphism.

8.1.5. We can talk about the full category
$$\mathrm{Whit}^{\mathrm{ext}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I) \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I),$$
where the "extended Whittaker condition" depends in the point in $(\mathbb{A}^1)^I$ (we think of this $(\mathbb{A}^1)^I$ as the variety of characters of $\mathbb{G}_a^I$), see Sect. 8.3.1.

Remark 8.1.6. The notation $\mathrm{Whit}^{\mathrm{ext}}$ (and the idea thereof) is borrowed from [Ga2, Sect. 8], where the *extended Whittaker category* is studied.

8.1.7. We can also consider the corresponding equivariant version
$$\mathrm{Whit}^{\mathrm{ext}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)^T \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)^T.$$

We will prove:

Proposition 8.1.8. *For $\mathcal{F} \in \mathrm{Whit}^{\mathrm{ext}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)^T$, supported off $\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^I$, the object*
$$(\overline{\mathsf{p}}/T)_! \circ (f/T)_!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G)$$
is Eisenstein.

Proposition 8.1.9. *For $\mathcal{F} \in \mathrm{Whit}^{\mathrm{ext}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)$, the map*
$$f_!(\mathcal{F}) \to f_*(\mathcal{F})$$
is an isomorphism.

It is clear that the combination of these two propositions implies that (7.9) is Eisenstein.

8.2. **Proof of Proposition 8.1.8.**

8.2.1. For a subset $J \subset I$, let
$$\mathbf{i}_J : (\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J) \subset (\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1)^I)$$
be the embedding of the corresponding stratum, so that $\mathbf{i}_I = \mathbf{j}$.

Denote by
$$\mathrm{Whit}^{\mathrm{part}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J) \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J)$$
and
$$\mathrm{Whit}^{\mathrm{part}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J)^T \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J)^T.$$
the corresponding subcategories, obtained by imposing the Whittaker-type equivariance condition.

Remark 8.2.2. The notation $\mathrm{Whit}^{\mathrm{part}}$ (and the idea thereof) is borrowed from [Ga2, Sect. 7].

8.2.3. We will show that the functor
$$(\overline{\mathsf{p}}/T)_! \circ (f/T)_! \circ (\mathbf{i}_J/T)_! : \mathrm{Whit}^{\mathrm{part}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)} \times (\mathbb{A}^1 - 0)^J)^T \to \mathrm{Bun}_G$$
factors through the subcategory generated by
$$\mathrm{Eis}_! : \mathrm{Shv}(\mathrm{Bun}_M) \to \mathrm{Shv}(\mathrm{Bun}_G),$$
where P is the standard parabolic corresponding to J.

[22] Recall that G was assumed semi-simple.

8.2.4. We stratify $\overline{\mathrm{Bun}}_{N,\rho(\omega)}$ by $(\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda$, $\lambda \in \Lambda^{\mathrm{pos}}$,

$$(\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \simeq \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)},$$

where $X^{(\lambda)} \to \mathrm{Bun}_T$ is the Abel-Jacobi map, shifted by $\rho(\omega_X)$. Let i_λ denote the corresponding locally closed embedding.

Consider the corresponding subcategories

$$\mathrm{Whit}^{\mathrm{part}}((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J) \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J)$$

and

$$\mathrm{Whit}^{\mathrm{part}}(\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J)^T \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J)^T.$$

We will show that the functor

$$(\overline{\mathsf{p}}/T)_! \circ (f/T)_! \circ (\mathbf{i}_J/T)_! \circ (i_\lambda/T)_! : \mathrm{Whit}^{\mathrm{part}}((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J)^T \to \mathrm{Bun}_G$$

factors as

$$\mathrm{Whit}^{\mathrm{part}}((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J)^T \to \mathrm{Bun}_M \overset{\mathrm{Eis}_!}{\to} \mathrm{Bun}_G\,.$$

8.2.5. Note that the map

$$\left((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J\right)/T \overset{(\overline{\mathsf{p}}/T)\circ(f/T)\circ(\mathbf{i}_J/T)\circ(i_\lambda/T)}{\longrightarrow} \mathrm{Bun}_G$$

factors as

$$\left((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J\right)/T \overset{f_P}{\longrightarrow} \mathrm{Bun}_P \overset{\mathsf{p}_P}{\longrightarrow} \mathrm{Bun}_G\,.$$

Moreover, we have a Cartesian diagram

$$\begin{array}{ccc}
\left((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J\right)/T & & \\
\simeq\big\downarrow & & \\
\left((\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)}) \times (\mathbb{A}^1-0)^J\right)/T & \overset{'\mathsf{q}_P}{\longrightarrow} & \left((\mathrm{Bun}_{B(M)} \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)}) \times (\mathbb{A}^1-0)^J\right)/T \\
f_P\big\downarrow & & \big\downarrow {'f_P} \\
\mathrm{Bun}_P & \overset{\mathsf{q}_P}{\longrightarrow} & \mathrm{Bun}_M \\
\mathsf{p}_P\big\downarrow & & \\
\mathrm{Bun}_G & &
\end{array}$$

and every object from $\mathrm{Whit}^{\mathrm{part}}\left((\overline{\mathrm{Bun}}_{N,\rho(\omega)})_\lambda \times (\mathbb{A}^1-0)^J\right)^T$ is isomorphic to the *-pullback by means of $'\mathsf{q}_P$ of an object in

$$\mathrm{Shv}\left(\left((\mathrm{Bun}_{B(M)} \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)}) \times (\mathbb{A}^1-0)^J\right)/T\right).$$

8.2.6. Now,

$$(\overline{\mathsf{p}}/T)_! \circ (f/T)_! \circ (\mathbf{i}_J/T)_! \circ (i_\lambda/T)_! \circ ('\mathsf{q}_P)^* \simeq \tag{8.3}$$
$$\simeq (\mathsf{p}_P)_! \circ (f_P)_! \circ ('\mathsf{q}_P)^* \simeq (\mathsf{p}_P)_! \circ (\mathsf{p}_P)^* \circ ('f_P)_! \simeq \mathrm{Eis}_! \circ ('f_P)_!$$

(up to a cohomological shift[23]), as required.

□[Proposition 8.1.8]

[23] Which is involved in the definition of $\mathrm{Eis}_!$.

Remark 8.2.7. We have obtained that the cone of (7.9) admits a canonical filtration indexed by $\emptyset \neq J \subset I$, where the subquotient corresponding to a given J in turn has a filtration indexed by $\lambda \in \Lambda^{\mathrm{pos}}$, and its subquotient corresponding to a given λ is given by

$$\mathrm{Eis}_!\Big(({}'f_P)_! \circ ({}'\mathfrak{q}_P)_* \circ (i_\lambda/T)^* \circ (\mathbf{i}_J/T)^* \circ (\mathbf{j}/T)_* \circ (\pi_{Z_G})_! \circ j_! \circ (\chi^I)^*(\exp^{\boxtimes \mathrm{I}})\Big), \tag{8.4}$$

(up to a cohomological shift).

One can describe the cone of (7.9) more conceptually, by combining the results of [Chen] and [Lin]. Namely, it has a canonical filtration indexed by the poset of proper parabolics with the associated graded corresponding to a given P being

$$\mathrm{Eis}_P^{\mathrm{enh}} \circ \mathrm{CT}_P^{\mathrm{enh}}(\mathrm{Poinc}_!^{\mathrm{Vac}}),$$

where we refer the reader to [Chen] for the "enhanced" notation.

Remark 8.2.8. The object

$$({}'f_P)_! \circ ({}'\mathfrak{q}_P)_* \circ (i_\lambda/T)^* \circ (\mathbf{i}_J/T)^* \circ (\mathbf{j}/T)_* \circ (\pi_{Z_G})_! \circ j_! \circ (\chi^I)^*(\exp^{\boxtimes \mathrm{I}}) \in \mathrm{Shv}(\mathrm{Bun}_M)$$

is closely related to (and can be algorithmically expressed via) the object

$$({}'f_P)_! \circ ({}'\mathfrak{q}_P)_* \circ (i_\lambda/T)^! \circ (\mathbf{i}_J/T)^! \circ (\mathbf{j}/T)_* \circ (\pi_{Z_G})_! \circ j_! \circ (\chi^I)^*(\exp^{\boxtimes \mathrm{I}}) \in \mathrm{Shv}(\mathrm{Bun}_M).$$

The computation of the latter objects is the main goal of the paper [AG2]. Namely, it says that this object identifies with

$$\mathrm{CT}_*(\mathrm{Poinc}_!^{\mathrm{Vac}}),$$

which in turn be calculated using Theorem 7.4.6.

8.3. **Proof of Proposition 8.1.9.**

8.3.1. The proof will fit into the following general paradigm:

Let $\mathcal{Y}$ be a stack acted on by a vector group V. Consider the corresponding category

$$\mathrm{Whit}(V^* \times \mathcal{Y}) \subset \mathrm{Shv}(V^* \times \mathcal{Y}).$$

Namely, this is the full subcategory consisting of objects $\mathcal{F} \in \mathrm{Whit}(V^* \times \mathcal{Y})$, equipped with an isomorphism

$$(\mathrm{id} \times \mathrm{act})^*(\mathcal{F}) \simeq (p_{1,2}^* \circ \mathrm{ev}^*(\exp)) \overset{*}{\otimes} p_{1,3}^*(\mathcal{F})$$

in $\mathrm{Shv}(V^* \times V \times \mathcal{Y})$ that restricts to the identity map[24] on $V^* \times \{0\} \times \mathcal{Y}$, where

- act denotes the action map $V \times \mathcal{Y} \to \mathcal{Y}$;
- ev denotes the evaluation map $V^* \times V \to \mathbb{G}_a$.

Note that the functor

$$\mathrm{Shv}(\mathcal{Y}) \overset{\mathrm{act}^*}{\to} \mathrm{Shv}(V \times \mathcal{Y}) \overset{\mathrm{Four}_V}{\longrightarrow} \mathrm{Shv}(V^* \times \mathcal{Y})$$

gives rise to an equivalence

$$\mathrm{Shv}(\mathcal{Y}) \overset{\sim}{\to} \mathrm{Whit}(V^* \times \mathcal{Y}).$$

[24]Since V is unipotent as a group, the full equivariance structure is actually a condition, which is equivalent to the simply-minded one above.

8.3.2. Let f denote the map $V^* \times \mathcal{Y} \to \mathcal{Y}$. We claim:

Proposition 8.3.3. *The natural transformation* $f_! \to f_*$ *becomes an isomorphism when evaluated an objects of* $\mathrm{Whit}(V^* \times \mathcal{Y})$.

Proof. It enough to prove the isomorphism after the (smooth) pullback by means of the map

$$V \times \mathcal{Y} \overset{\mathrm{act}}{\to} \mathcal{Y}.$$

This reduces us to the case when $\mathcal{Y}$ has the form $V \times \mathcal{Z}$ with V acting on the first factor.

In this case, the equivalence

$$\mathrm{Shv}(V \times \mathcal{Z}) \to \mathrm{Whit}(V^* \times V \times \mathcal{Z})$$

is given by

$$\mathcal{F} \mapsto p_{1,3}^*(\mathrm{Four}_{\mathcal{Z}}(\mathcal{F})) \overset{*}{\otimes} p_{1,2}^*(\mathrm{mult}^*(\exp)).$$

The operations

$$\mathcal{G} \in \mathrm{Shv}(V^* \times \mathcal{Z}) \rightsquigarrow f_! \left(p_{1,3}^*(\mathcal{G}) \overset{*}{\otimes} p_{1,2}^*(\mathrm{mult}^*(\exp)\right) \text{ and } f_* \left(p_{1,3}^*(\mathcal{G}) \overset{*}{\otimes} p_{1,2}^*(\mathrm{mult}^*(\exp)\right)$$

are the !- and *- versions of the functor

$$\mathrm{Four}_{\mathcal{Z}} : \mathrm{Shv}(V^* \times \mathcal{Z}) \to \mathrm{Shv}(V \times \mathcal{Z}).$$

Now, it is well-know that the natural transformation

$$\mathrm{Four}_{!,\mathcal{Z}} \to \mathrm{Four}_{*,\mathcal{Z}}$$

is an isomorphism.

□

8.3.4. We apply the above paradigm as follows. Cover $\overline{\mathrm{Bun}}_{N,\rho(\omega)}$ by open substacks

$$\overline{\mathrm{Bun}}_{N,\rho(\omega),\mathrm{good\,at\,x}}, \quad x \in X,$$

where we require that the generalized B-reduction be *non-degenerate* at x. It is enough to show that the map $f_!(\mathcal{F}) \to f_*(\mathcal{F})$ restricts to an isomorphism over every

$$\overline{\mathrm{Bun}}_{N,\rho(\omega),\mathrm{good\,at\,x}}.$$

8.3.5. Consider the corresponding stack

$$\overline{\mathrm{Bun}}^{\mathrm{level}_x}_{N,\rho(\omega),\mathrm{good\,at\,x}},$$

see [Ga3, Sect. 4.4.2].

The stack $\overline{\mathrm{Bun}}^{\mathrm{level}_x}_{N,\rho(\omega),\mathrm{good\,at\,x}}$ is acted on by the group ind-scheme $\mathfrak{L}(N_{\rho(\omega)})_x$. Let

$$\chi_x^I : \mathfrak{L}(N_{\rho(\omega)})_x \to \mathbb{G}_a^I$$

denote the canonical character.

8.3.6. Let $N' \subset \mathfrak{L}(N_{\rho(\omega)})_x$ be a sufficiently large subgroup. Let

$$\overset{\circ}{N}{}' := \ker(\chi_x^I|_{N'}).$$

Set

$$\mathcal{Y} := \overline{\mathrm{Bun}}^{\mathrm{level}_x}_{N,\rho(\omega),\mathrm{good\,at\,x}}/\overset{\circ}{N}{}'.$$

This is a stack locally of finite type, which carries an action of $V := \mathbb{G}_a^I$. Finally, we note that the category $\mathrm{Whit}(V^* \times \mathcal{Y})$ identifies with the category

$$\mathrm{Whit}^{\mathrm{ext}}(\overline{\mathrm{Bun}}_{N,\rho(\omega),\mathrm{good\,at\,x}} \times (\mathbb{A}^1)^I),$$

where we think of $(\mathbb{A}^1)^I$ as V^*.

□[Proposition 8.1.9]

8.4. **The Kirillov model.** As a preparation to the proof of Theorem 7.5.4 in the setting where the exponential sheaf does not exist, we discuss the fomalism of *Kirillov* (as opposed to Whittaker) models.

8.4.1. Let $\mathcal{Y}$ be a stack equipped with an action of $\mathbb{G}_a^I$. For every subset $J \subset I$, denote by

$$\operatorname{Shv}(\mathcal{Y})_{J\text{-cl}} \overset{(\bar{\mathbf{i}}_J)_! = (\bar{\mathbf{i}}_J)_*}{\hookrightarrow} \operatorname{Shv}(\mathcal{Y})$$

the embedding of the full subcategory $\operatorname{Shv}(\mathcal{Y})^{\mathbb{G}_a^J}$.

The above functor admits a left and right adjoints, denoted $(\bar{\mathbf{i}}_J)^*$ and $(\bar{\mathbf{i}}_J)^!$, explicitly given by $\operatorname{Av}_!^{\mathbb{G}_a^J}$ and $\operatorname{Av}_*^{\mathbb{G}_a^J}$, respectively.

8.4.2. Let $\operatorname{Shv}(\mathcal{Y})_J$ be the quotient of $\operatorname{Shv}(\mathcal{Y})_{J\text{-cl}}$ obtained by modding out with respect to all $\operatorname{Shv}(\mathcal{Y})_{J'\text{-cl}}$ with $J' \supset J$. Denote by $(\mathbf{j}_J)^* = (\mathbf{j}_J)^!$ the projection

$$\operatorname{Shv}(\mathcal{Y})_{J\text{-cl}} \twoheadrightarrow \operatorname{Shv}(\mathcal{Y})_J.$$

This projection admits left and right adjoints, denoted $(\mathbf{j}_J)_!$ and $(\mathbf{j}_J)_*$, respectively.

8.4.3. Denote

$$(\mathbf{i}_J)_! := (\bar{\mathbf{i}}_J)_! \circ (\mathbf{j}_J)_! \text{ and } (\mathbf{i}_J)_* := (\bar{\mathbf{i}}_J)_* \circ (\mathbf{j}_J)_*, \quad \operatorname{Shv}(\mathcal{Y})_J \to \operatorname{Shv}(\mathcal{Y}).$$

These functors admit right and left adjoints, given by

$$(\mathbf{i}_J)^! := (\mathbf{j}_J)^! \circ (\bar{\mathbf{i}}_J)^! \text{ and } (\mathbf{i}_J)^* := (\mathbf{j}_J)^* \circ (\bar{\mathbf{i}}_J)^*,$$

respectively.

Thus, we obtain a stratification of $\operatorname{Shv}(\mathcal{Y})$ with subquotients $\operatorname{Shv}(\mathcal{Y})_J$.

8.4.4. In particular, we have the "open stratum" corresponding to $J = \emptyset$.

Note that the essential image of $(\mathbf{j}_\emptyset)_!$ (resp., $(\mathbf{j}_\emptyset)_*$) consists of objects for which the !- (resp., *-) averaging for any coordinate copy of $\mathbb{G}_a \subset \mathbb{G}_a^I$ vanishes.

On general categorical grounds, we have a natural transformation

$$(\mathbf{j}_\emptyset)_! \to (\mathbf{j}_\emptyset)_*. \tag{8.5}$$

8.4.5. Let now T be a torus equipped with a surjection onto $\mathbb{G}_m^I$. Denote

$$Z_G := \ker(T \to \mathbb{G}_m^I).$$

Assume that the action of $\mathbb{G}_a^I$ on $\mathcal{Y}$ can be extended to an action of $T \ltimes \mathbb{G}_a^I$.

Then the discussion in Sects. 8.4.1-8.4.3 is applicable to $\operatorname{Shv}(\mathcal{Y}/T)$. In particular, we obtain the sub/quotient categories

$$\operatorname{Shv}(\mathcal{Y}/T)_{J\text{-cl}} \text{ and } \operatorname{Shv}(\mathcal{Y}/T)_J,$$

and the corresponding functors

$$(\mathbf{i}_J/T)_!, (\mathbf{i}_J/T)_*, (\mathbf{i}_J/T)^!, (\mathbf{i}_J/T)^*, \text{etc.}$$

8.4.6. For the next few subsection we will assume that we are in the situation in which the exponential sheaf exists, and we will explain the connection with the Whittaker picture.

Consider the full subcategories

$$\mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I) \subset \mathrm{Shv}(\mathcal{Y} \times (\mathbb{A}^1)^I)$$

and

$$\mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I)^T \subset \mathrm{Shv}(\mathcal{Y} \times (\mathbb{A}^1)^I)^T,$$

cf. Sect. 8.1.5.

Consider the corresponding functors

$$\mathbf{j}_! : \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I) \to \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I)$$

and

$$(\mathbf{j}/T)_! : \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T \to \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I)^T.$$

Recall now that according to Sect. 8.3.1, we have an equivalence

$$\mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I) \simeq \mathrm{Shv}(\mathcal{Y}),$$

explicitly given by !- (equivalently, *) direct image along the projection $\mathcal{Y} \times (\mathbb{A}^1)^I \to \mathcal{Y}$.

This induces an equivalence

$$\mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I)^T \simeq \mathrm{Shv}(\mathcal{Y}/T). \tag{8.6}$$

The following is an elementary verification:

Lemma 8.4.7. *The equivalence* (8.6) *fits onto the commutative diagram*

$$\begin{array}{ccc} \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T & \xrightarrow{(\mathbf{j}/T)_!} & \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1)^I)^T \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}(\mathcal{Y}/T)_\emptyset & \xrightarrow{(\mathbf{j}_\emptyset/T)_!} & \mathrm{Shv}(\mathcal{Y}/T). \end{array}$$

8.4.8. Note that using the element $(1, ..., 1) \in (\mathbb{A}^1)^I$, we can identify

$$\mathrm{Shv}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T) \simeq \mathrm{Shv}(\mathcal{Y}/Z_G)$$

Under this identification, the subcategory

$$\mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T \subset \mathrm{Shv}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T)$$

corresponds to

$$\mathrm{Whit}(\mathcal{Y}/Z_G) \subset \mathrm{Shv}(\mathcal{Y}/Z_G).$$

We claim:

Lemma 8.4.9. *The equivalence*

$$\mathrm{Whit}(\mathcal{Y}/Z_G) \simeq \mathrm{Whit}^{\mathrm{ext}}(\mathcal{Y} \times (\mathbb{A}^1 - 0)^I)^T \simeq \mathrm{Shv}(\mathcal{Y}/T)_\emptyset$$

is given by the composition

$$\mathrm{Whit}(\mathcal{Y}/Z_G) \hookrightarrow \mathrm{Shv}(\mathcal{Y}/Z_G) \overset{!-pushforward}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}/T),$$

whose image lands in $\mathrm{Shv}(\mathcal{Y}/T)_\emptyset \subset \mathrm{Shv}(\mathcal{Y}/T)$.

8.4.10. We now return to the situation when the exponential sheaf does not necessarily exist. We let $\mathcal{Y} = \mathbb{G}_a^I$, equipped with a natural action of $T \ltimes \mathbb{G}_a^I$.

Note that due to monodromicity, the essential image of the embedding

$$\mathrm{Shv}(\mathbb{G}_a^I/T)_\emptyset \overset{(\mathbf{j}_\emptyset)_!}{\hookrightarrow} \mathrm{Shv}(\mathbb{G}_a^I/T)$$

consists of objects whose !-restrictions to

$$\mathbb{G}_a^J \subset \mathbb{G}_a^I, \quad J \neq I$$

are 0. I.e., these are objects that are *-extended from $(\mathbb{G}_a - 0)^I \overset{\jmath}{\hookrightarrow} \mathbb{G}_a^I$.

Similarly, the essential image of the embedding

$$\mathrm{Shv}(\mathbb{G}_a^I/T)_\emptyset \overset{(\mathbf{j}_\emptyset)_*}{\hookrightarrow} \mathrm{Shv}(\mathbb{G}_a^I/T)$$

consists of objects whose *-restrictions to

$$\mathbb{G}_a^J \subset \mathbb{G}_a^I, \quad J \neq I$$

are 0. I.e., these are objects that are !-extended from $(\mathbb{G}_a - 0)^I \overset{\jmath}{\hookrightarrow} \mathbb{G}_a^I$.

We obtain that the category $\mathrm{Shv}(\mathbb{G}_a^I/T)_\emptyset$ identifies with

$$\mathrm{Shv}((\mathbb{G}_a - 0)^I/T) \simeq \mathrm{Shv}(\mathrm{pt}/Z_G)$$

in two different ways[25].

In what follows we will use the identification

$$\mathrm{Shv}(\mathbb{G}_a^I/T)_\emptyset \overset{(\mathbf{j}_\emptyset)_!}{\hookrightarrow} \mathrm{Shv}(\mathbb{G}_a^I/T) \overset{(\jmath/T)^*}{\to} \mathrm{Shv}((\mathbb{G}_a - 0)^I/T) \simeq \mathrm{Shv}(\mathrm{pt}/Z_G). \tag{8.7}$$

8.4.11. Denote

$$\exp_\emptyset^{\boxtimes I}/T := R_{Z_G}[-r] \in \mathrm{Shv}(\mathrm{pt}/Z_G) \overset{(8.7)}{\simeq} \mathrm{Shv}(\mathbb{G}_a^I/T)_\emptyset,$$

where:

- $R_{Z_G} \in \mathrm{Shv}(\mathrm{pt}/Z_G)$ is the !-direct image of $\mathsf{e} \in \mathrm{Vect} = \mathrm{Shv}(\mathrm{pt})$ along $\mathrm{pt} \to \mathrm{pt}/Z_G$;
- $r = |I|$ is the semi-simple rank of G.

In terms of the above identifications, we have

$$\exp_!^{\boxtimes I}/T = (\mathbf{j}_\emptyset)_!(\exp_\emptyset^{\boxtimes I}/T),$$

Set

$$\exp_*^{\boxtimes I}/T := (\mathbf{j}_\emptyset)_*(\exp_\emptyset^{\boxtimes I}/T).$$

Thus, explicitly,

$$\exp_!^{\boxtimes I}/T \simeq (\jmath/T)_*(R_{Z_G})[-r]$$

and

$$\exp_*^{\boxtimes I}/T \simeq (\jmath/T)_!(R_{Z_G})[-r].$$

Note that the natural transformation (8.5) gives rise to a map

$$\exp_!^{\boxtimes I}/T \to \exp_*^{\boxtimes I}/T. \tag{8.8}$$

In other words, this is a map

$$(\jmath/T)_*(R_{Z_G})[-r] \to (\jmath/T)_!(R_{Z_G})[-r], \tag{8.9}$$

note the direction of the arrow!

[25]They differ by a cohomological shift.

8.4.12. *Examples.* Let us analyze the behavior of the map (8.9) explicitly.

. We consider the pullback of the map (8.9) to $\mathbb{G}_a^I$ itself. Thus, we are dealing with the map

$$\jmath_*(\pi_!(\mathsf{e}_T)) \to \jmath_!(\pi_!(\mathsf{e}_T)), \tag{8.10}$$

where $\pi : T \to \mathbb{G}_m^I$.

Assume first that $T \to \mathbb{G}_m^I$ is an isomorphism. Then both sides of (8.10), shifted cohomologically by $[r]$, are perverse, and the cosocle (resp., socle) of $\jmath_*(\mathsf{e}_{\mathbb{G}_m^I})[r]$ (resp., of $\jmath_!(\mathsf{e}_{\mathbb{G}_m^I})[r]$) is the δ-function at 0, i.e., $\underline{\mathsf{e}}_{\mathrm{pt}}$.

With these identifications, the map (8.9) is

$$\jmath_*(\mathsf{e}_{\mathbb{G}_m^I})[r] \to \underline{\mathsf{e}}_{\mathrm{pt}} \to \jmath_*(\mathsf{e}_{\mathbb{G}_m^I})[r].$$

In the general case, both sides in (8.10) have additional direct factors, given by Kummer sheaves corresponding to non-zero characters of (the finite group) Z_G. The map (8.10) is the natural isomorphism on these factors.

8.5. **The case when there is no exponential sheaf.** In this subsection we will treat the case of Theorem 7.5.4 when we work either over a field of characteristic 0 or in mixed characteristic.

8.5.1. We adapt the formalism of Sect. 8.4 to $\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}$, using the method of Sects. 8.3.4-8.3.6.

Thus, we obtain a stratification of the category $\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})$ (resp., $\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)$) by sub/quotient categories $\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_J$ (resp., $\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)_J$), $J \subset I$, and similarly for $\mathrm{Bun}_{N,\rho(\omega_X)}$

8.5.2. Consider the object

$$\exp_{\chi^I,\emptyset}/T := (\chi^I/T)^*(\exp_\emptyset^{\boxtimes I}/T) \in \mathrm{Shv}(\mathrm{Bun}_{N,\rho(\omega_X)}/T)_\emptyset.$$

First, we note:

Lemma 8.5.3. *For $J \neq 0$, the projection*

$$\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}) \overset{(\mathbf{j}_\emptyset)^*=(\mathbf{j}_\emptyset)^!}{\twoheadrightarrow} \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\emptyset$$

annihilates objects supported on $\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} - \mathrm{Bun}_{N,\rho(\omega_X)}$.

Hence, we obtain that $\exp_{\chi^I,\emptyset}/T$ uniquely extends to an object of $\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)_\emptyset$; we denote it by

$$\overline{\exp}_{\chi^I,\emptyset}/T.$$

8.5.4. Unwinding the constructions, we obtain that the objects $\mathrm{Poinc}_!^{\mathrm{Vac}}$ and $\mathrm{Poinc}_*^{\mathrm{Vac}}$ are

$$(\overline{\mathsf{p}}/T)_! \circ (\mathbf{j}_\emptyset)_!(\overline{\exp}_{\chi^I,\emptyset}/T) \text{ and } (\overline{\mathsf{p}}/T)_* \circ (\mathbf{j}_\emptyset)_*(\overline{\exp}_{\chi^I,\emptyset}/T),$$

respectively, where $(\overline{\mathsf{p}}/T)_! \simeq (\overline{\mathsf{p}}/T)_*$, since the map $\overline{\mathsf{p}}$ is proper.

Moreover, the map (7.9) is induced by the natural transformation (8.5).

Hence, in order to prove Theorem 7.5.4, it suffices to show that for $J \neq \emptyset$, the essential image of the functor

$$\mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)_{J\text{-cl}} \overset{(\bar{\mathbf{i}}_J)_!=(\bar{\mathbf{i}}_J)_*}{\longrightarrow} \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T) \overset{(\overline{\mathsf{p}}/T)_!}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \tag{8.11}$$

lies in the essential image of the functor

$$\mathrm{Shv}(\mathrm{Bun}_M) \overset{\mathrm{Eis}_!}{\to} \mathrm{Shv}(\mathrm{Bun}_G),$$

where $P \twoheadrightarrow M$ are the standard Levi and parabolic corresponding to J.

This is done by a manipulation similar to the one used in the proof of Proposition 8.1.8. We elaborate on this below.

8.5.5. We stratify $\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}$ by

$$(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda \overset{i_\lambda}{\hookrightarrow} \overline{\mathrm{Bun}}_{N,\rho(\omega_X)},$$

and consider the corresponding categories

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T)_{J\text{-cl}} \overset{(\bar{\mathfrak{i}}_J)_!=(\bar{\mathfrak{i}}_J)_*}{\longrightarrow} \mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T).$$

We will show that the composition

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T)_{J\text{-cl}} \overset{(i_\lambda)_!}{\hookrightarrow} \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)_{J\text{-cl}} \overset{(\bar{\mathfrak{i}}_J)_!=(\bar{\mathfrak{i}}_J)_*}{\longrightarrow} \to \mathrm{Shv}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T) \overset{(\bar{\mathfrak{p}}/T)_!}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \tag{8.12}$$

factors as

$$\mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T)_{J\text{-cl}} \to \mathrm{Shv}(\mathrm{Bun}_M) \overset{\mathrm{Eis}_!}{\to} \mathrm{Shv}(\mathrm{Bun}_G),$$

where P is the standard parabolic corresponding to $J \subset I$.

This follows from the Cartesian diagram

$$\begin{array}{ccc}
(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T & & \\
\sim\downarrow & & \\
(\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)})/T & \overset{'\mathsf{q}_P}{\longrightarrow} & (\mathrm{Bun}_{B(M)} \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)})/T \\
\downarrow & & \downarrow \\
\mathrm{Bun}_P & \overset{\mathsf{q}_P}{\longrightarrow} & \mathrm{Bun}_M \\
\mathsf{p}_P\downarrow & & \\
\mathrm{Bun}_G, & &
\end{array}$$

using the following observation:

Lemma 8.5.6. *The functor of *-pullback along* $'\mathsf{q}_P$ *gives rise to an equivalence*

$$\mathrm{Shv}((\mathrm{Bun}_{B(M)} \underset{\mathrm{Bun}_T}{\times} X^{(\lambda)})/T) \to \mathrm{Shv}((\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})_\lambda/T)_{J\text{-cl}}.$$

□[Theorem 7.5.4]

9. Langlands functor and Eisenstein series

The goal of this section is to prove Theorem 1.4.6.

Remark 9.0.1. The proof that we will give applies in any sheaf-theoretic context (e.g., de Rham and Betti). We note, however, that the construction of the isomorphism of functors stated in Theorem 1.4.6 is a priori different from that in [GLC3, Theorem 14.2.2].

The reason that we give a (different) proof here is that some ingredients of the proof in [GLC3] (specifically, the semi-infinite geometric Satake) have only been developed in the de Rham context.

It is a good (but potentially quite involved) exercise to show that the two isomorphisms actually agree.

9.1. **Reducing to a statement about $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}$.** This step is parallel to [GLC3, Sect. 14.2.3].

9.1.1. Denote

$$\mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}} := \Upsilon^\vee_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}} \circ \mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}} \circ \Xi_{\mathrm{LS}^{\mathrm{restr}}_{\check{M}}}, \quad \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

In other words,

$$\mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}} = (\mathsf{p}^{-,\mathrm{spec}})_* \circ (\mathsf{q}^{-,\mathrm{spec}})^*;$$

note that map $\mathsf{p}^{-,\mathrm{spec}}$ is schematic, so the functor $(\mathsf{p}^{-,\mathrm{spec}})_*$ is well-behaved.

9.1.2. We claim that in order to construct the datum of commutativity for the diagram (1.6), it is enough to do so for the diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}^{\mathrm{restr}}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{(N^-_P)_{\rho_P(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}]{} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}). \end{array} \tag{9.1}$$

Indeed, to prove the commutativity of (1.6), it is enough to show that the two circuits are isomorphic when evaluated on compact objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Note that diagram (9.1) is obtained from diagram (1.6) by composing both circuits with

$$\Upsilon^\vee_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

Since the functor $\Upsilon^\vee_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}}$ is fully faithful on the eventually coconnective subcategory, it suffices to show that both circuits in (1.6) send compact objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to eventually coconnective objects in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

9.1.3. We first show this for the anti-clockwise circuit.

The functor $\mathrm{Eis}^-_!$ preserves compactness, and hence so does its translated version $\mathrm{Eis}^-_{!,\rho_P(\omega_X)}$. Hence, the required assertion follows from Theorem 1.1.10.

9.1.4. We now consider the clockwise circuit. The top horizontal arrow sends compact objects to eventually coconnective objects again by Theorem 1.1.10 (applied to M).

Finally, the functor $\mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}$ has a bounded cohomological amplitude on the left, since the morphism $\mathsf{q}^{-,\mathrm{spec}}$ is quasi-smooth.

9.2. **Method of proof.** The proof will be a geometric counterpart of the computation of Whittaker coefficients of Eisenstein series, coupled with (the parabolic version of) the theory developed in [BG2] and [FH].

9.2.1. Recall that the functor $\mathrm{Eis}^-_!$ extends to a $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-linear functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) \to \mathrm{Shv}(\mathrm{Bun}_G), \tag{9.2}$$

see the proof of Proposition 1.4.2. We will denote the functor in (9.2) by $\mathrm{Eis}^{-,\mathrm{part.enh}}_!$.

We will denote by $\mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}$ the precomposition of $\mathrm{Eis}^{-,\mathrm{part.enh}}_!$ with the corresponding translation functor (this is well-defined as the action of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$ commutes with central translations).

Remark 9.2.2. The functor $\mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}$ corresponds under the Langlands functor to the functor $\mathrm{Eis}^{-,\mathrm{spec},\mathrm{part.enh}}$ studied in [GLC3, Sect. 12.6-7].

9.2.3. In order to construct the commutativity datum for (9.1), it suffices to do so for the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) & \xrightarrow{\mathrm{Id}\otimes \mathbb{L}^{\mathrm{restr}}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \\ {\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}[\delta_{(N^-_P)_{\rho_P(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle (\mathsf{q}^{-,\mathrm{spec}})_*} \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}]{} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}). \end{array} \tag{9.3}$$

Since both circuits in (9.3) are $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-linear, in order to construct the commutativity datum for (9.3), it is enough to do so for its composition with the functor $\Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}, -)$.

Thus, we need to construct the datum of commutativity for the diagram

$$
\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) & \xrightarrow{\mathrm{Id}\otimes \mathbb{L}^{\mathrm{restr}}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \\
\Big\downarrow {\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}[\delta_{(N^-_P)_{\rho_P(\omega_X)}}]} & & \Big\downarrow {\scriptstyle \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-},-)} \\
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow[\mathrm{coeff}^{\mathrm{Vac}}_G]{} & \mathrm{Vect}\,.
\end{array}
\tag{9.4}
$$

9.2.4. Denote

$$\Omega^{\mathrm{glob}} := \mathsf{q}^{-,\mathrm{spec}}_*(\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}}) \in \mathrm{ComAlg}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})).$$

The morphism $\mathsf{q}^{-,\mathrm{spec}}$ is "as good as affine" (see [GLC2, Sect. 12.7.5] for what this means). Hence, we can identify

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{P}^-}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) \simeq \Omega^{\mathrm{glob}}\text{-mod}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)).$$

The datum of the functor $\mathrm{Eis}^{-,\mathrm{part.enh}}_!$ can be interpreted as the action of Ω^{glob}, viewed as a monad on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$, on the functor $\mathrm{Eis}^-_!$.

Thus, the datum of commutativity of (9.4) can be stated as an isomorphism of functors

$$\mathrm{coeff}^{\mathrm{Vac}}_G \circ \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{(N^-_P)_{\rho_P(\omega_X)}}] \simeq \mathrm{coeff}^{\mathrm{Vac}}_M \circ (\Omega^{\mathrm{glob}} \star -) \tag{9.5}$$

as functors

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M) \to \mathrm{Vect},$$

acted on by the monad Ω^{glob}. In the above formula, $(-)\star(-)$ denotes the action of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$.

In what follows we will construct an isomorphism (9.5); its compatibility with the action of Ω^{glob} would follow by unwinding the construction of [BG2] and [FH][26], see Remark 9.3.13.

9.2.5. Recall the category $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$, which acts on $\mathrm{Shv}(\mathrm{Bun}_M)$. Its action of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_M)$ factors through the localization functor

$$\mathrm{Loc}^{\mathrm{spec,restr}}_{\check{M}} : \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}}),$$

see [AGKRRV1, Sect. 12.7.1].

Let $\Omega^{\mathrm{loc}} \in \mathrm{Rep}(\check{M})$ be the commutative algebra

$$\mathrm{C}^\cdot_{\mathrm{Chev}}(\check{\mathfrak{n}}^-_P).$$

We attach to it the commutative factorization algebra

$$\mathrm{Fact}(\Omega^{\mathrm{loc}}) \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}},$$

which we view as a (commutative) algebra object with respect to the (symmetric) monoidal structure on $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$, see [GLC2, Sect. B.10.4].

By [GLC3, Sect. 12.3.5], we have

$$\Omega^{\mathrm{glob}} \simeq \mathrm{Loc}^{\mathrm{spec,restr}}_{\check{G}}(\mathrm{Fact}(\Omega^{\mathrm{loc}})),$$

as commutative algebra objects in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{M}})$.

[26]More precisely, this follows by combining the Koszul duality statement in the proof of Corollary 6.4.1.2 with the proof of Proposition 4.5.4.1 in [FH].

9.2.6. We can view $\text{Fact}(\Omega^{\text{loc}})$ as a monad acting on $\text{Shv}(\text{Bun}_M)$. We will prove the following generalization of (9.5):

Theorem 9.2.7. *There exists a canonical isomorphism*

$$\text{coeff}_G^{\text{Vac}} \circ \text{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{(N_P^-)_{\rho_P(\omega_X)}}] \simeq \text{coeff}_M^{\text{Vac}} \circ (\text{Fact}(\Omega^{\text{loc}}) \star (-))$$

as functors $\text{Shv}(\text{Bun}_M) \to \text{Vect}$*, where* $(-) \star (-)$ *denotes the monoidal action of* $\text{Rep}(\check{M})_{\text{Ran}}$ *on* $\text{Shv}(\text{Bun}_G)$.

Remark 9.2.8. One can view formula (9.5) as a geometric counterpart of the classical computation of the Whittaker coefficient of Eisenstein series.

The latter says that the Whittaker coefficient of Eisenstein series of an automorphic function on M equals the Whittaker coefficient of a particular Hecke operator applied to that automorphic function.

The Hecke functor $\text{Fact}(\Omega^{\text{loc}})$ is the geometric counterpart of that classical Hecke functor.

Remark 9.2.9. Theorem 9.2.7 is equivalent to a (particular case of) [GLC3, Corollary 10.1.5], and its proof is parallel to that of [GLC3, Theorem 10.1.2] in Sect. 10.3 of *loc. cit.*

The essential difference[27], however, is that our key computational ingredient here is Proposition 9.3.12, whereas in [GLC3] it is Proposition 10.6.8 of *loc. cit.*, which uses the local theory developed in that paper, specifically Corollary 2.5.2 in *loc. cit.*.

9.3. **Proof of Theorem 9.2.7.** We will give a proof when $\text{char}(k)$ is positive, i.e., when the Artin-Schreier sheaf exists; the proof in the characteristic zero case is completely parallel: one replaces the Artin-Schreier sheaf by the procedure of [GLC1, Sect. 3.3].

9.3.1. Denote

$$\text{Bun}_{N,\rho(\omega_X)} := \text{Bun}_B \underset{\text{Bun}_T}{\times} \{\rho(\omega_X)\},$$

and denote by

$$\chi : \text{Bun}_{N,\rho(\omega_X)} \to \mathbb{G}_a$$

that map

$$\text{Bun}_{N,\rho(\omega_X)} \overset{\chi^I}{\to} \mathbb{G}_a^I \overset{\text{sum}}{\to} \mathbb{G}_a.$$

Let p denote the projection

$$\text{Bun}_{N,\rho(\omega_X)} \to \text{Bun}_G .$$

Recall that the functor $\text{coeff}^{\text{Vac}}$ is by defintion

$$\text{C}^\cdot(\text{Bun}_{N,\rho(\omega_X)}, \mathsf{p}^!(-) \overset{!}{\otimes} \chi^!(\exp_\omega)), \tag{9.6}$$

where $\exp_\omega$ denotes the Artin-Schreier sheaf on $\mathbb{G}_a$, normalized so that it behaves multiplicatively with respect to the !-pullback.

9.3.2. Recall that the functor $\text{Eis}^-_!$ is defined by

$$(\mathsf{p}^-)_! \circ (\mathsf{q}^-)^*(-)[\text{dim. rel}], \tag{9.7}$$

where:

- $\mathsf{p}^- : \text{Bun}_{P^-} \to \text{Bun}_G$;
- $\mathsf{q}^- : \text{Bun}_{P^-} \to \text{Bun}_M$;
- dim. rel is the dimension of Bun_{P^-} over Bun_M (it depends on the connected component of Bun_M).

[27] Alluded to in Remark 9.0.1.

9.3.3. Let

$$\begin{array}{ccc} \widetilde{\mathrm{Bun}}_{P^-} & \xleftarrow{j} & \mathrm{Bun}_{P^-} \\ {\scriptstyle \widetilde{\mathsf{p}}^-}\big\downarrow & & \\ \mathrm{Bun}_G & & \end{array}$$

denote Drinfeld's relative compactification of Bun_{P^-} along p^-. Denote by $\widetilde{\mathsf{q}}^-$ the map $\widetilde{\mathrm{Bun}}_{P^-} \to \mathrm{Bun}_M$.

Note that the ULA property of the object $j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}) \in \mathrm{Shv}(\widetilde{\mathrm{Bun}}_{P^-})$ with respect to $\widetilde{\mathsf{q}}^-$ (see [BG1, Theorem 5.1.5]) implies that the natural transformation

$$j_! \circ (\widetilde{\mathsf{q}}^-)^*(-) \simeq j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}) \overset{*}{\otimes} (\widetilde{\mathsf{q}}^-)^*(-) \to j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}) \overset{!}{\otimes} (\widetilde{\mathsf{q}}^-)^!(-)[2\dim(\mathrm{Bun}_M)]$$

is an isomorphism.

Hence, we can rewrite

$$\mathrm{Eis}^-_!(-) \simeq (\widetilde{\mathsf{p}}^-)_*\left((\widetilde{\mathsf{q}}^-)^!(-) \overset{!}{\otimes} j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}})\right)[2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel}]. \tag{9.8}$$

9.3.4. Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-} & & \\ & {\scriptstyle '\widetilde{\mathsf{p}}^-}\swarrow & & \searrow{\scriptstyle '\mathsf{p}} & \\ \mathrm{Bun}_{N,\rho(\omega_X)} & & & & \widetilde{\mathrm{Bun}}_{P^-} \\ & {\scriptstyle \mathsf{p}}\searrow & & \swarrow{\scriptstyle \widetilde{\mathsf{p}}^-} & \\ & & \mathrm{Bun}_G, & & \end{array}$$

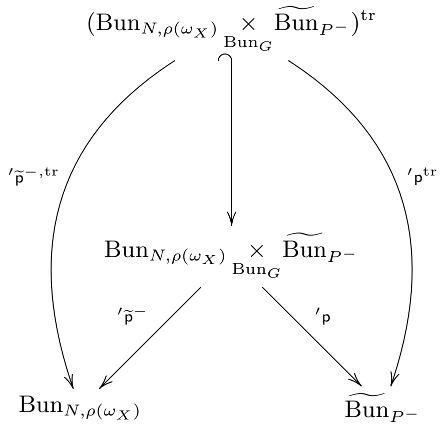
By base change and projection formula, we obtain:

$$\mathrm{coeff}^{\mathrm{Vac}}_G \circ \mathrm{Eis}^-_!(-) \simeq \tag{9.9}$$
$$\simeq \mathrm{C}^{\cdot}\left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-}, \left(('\mathsf{p})^! \circ ((\widetilde{\mathsf{q}}^-)^!(-) \overset{!}{\otimes} j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}))\right) \overset{!}{\otimes} \left(('\widetilde{\mathsf{p}}^-)^! \circ \chi^!(\exp_\omega)\right)\right)$$
$$[2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel}].$$

9.3.5. Let

$$(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}} \subset \mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-}$$

be the open substack corresponding to the condition that the N-reduction and the generalized P^--reduction of the G-bundle are transversal at the generic point of the curve:



As in [GLC3, Lemma 10.6.3], we have:

Lemma 9.3.6. *The natural transformation*

$$\mathrm{C}^\cdot\left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-}, \big(('\mathsf{p})^! \circ ((\widetilde{\mathsf{q}}^-)^!(-) \overset{!}{\otimes} j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}))\big) \overset{!}{\otimes} \Big(('\widetilde{\mathsf{p}}^-)^! \circ \chi^!(\exp_\omega)\Big)\right) \to$$
$$\mathrm{C}^\cdot\left((\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}}, \big(('\mathsf{p}^{\mathrm{tr}})^! \circ ((\widetilde{\mathsf{q}}^-)^!(-) \overset{!}{\otimes} j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}))\big) \overset{!}{\otimes} \Big(('\widetilde{\mathsf{p}}^{-,\mathrm{tr}})^! \circ \chi^!(\exp_\omega)\Big)\right)$$

is an isomorphism.

Hence, we obtain that the expression in (9.9) can be rewritten as

$$\mathrm{C}^\cdot\left((\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}}, \big(('\mathsf{p}^{\mathrm{tr}})^! \circ ((\widetilde{\mathsf{q}}^-)^!(-) \overset{!}{\otimes} j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}))\big) \overset{!}{\otimes} \Big(('\widetilde{\mathsf{p}}^{-,\mathrm{tr}})^! \circ \chi^!(\exp_\omega)\Big)\right) [2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel}]. \tag{9.10}$$

9.3.7. Recall the (parabolic) Zastava space

$$\mathrm{Zast} := (\mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}},$$

which is the open substack of $\mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-}$, corresponding to the condition that the P-reduction and the generalized P^--reduction are transversal at the generic point of the curve, see [BFGM, Sect. 2.2 and Proposition 3.2].

The stack Zast is endowed with a map

$$\mathfrak{s} : \mathrm{Zast} \to \mathrm{Hecke}(M)_{\mathrm{Ran}'},$$

where:

- Ran' is the sheafification of the Ran space in the topology of finite surjective maps;
- $\mathrm{Hecke}(M)_{\mathrm{Ran}'}$ is the corresponding version of the Hecke stack for M;

Remark 9.3.8. In the formulation in [BFGM], the map $\mathfrak{s}$ rather goes to a version of $\mathrm{Hecke}(M)$ over the space of colored divisors on X, which is the union of schemes of the form

$$X^{(\underline{n})} := \prod_i X^{(n_i)}, \quad n_i \in \mathbb{Z}^{\geq 0},$$

where i runs over (a subset of) the Dynkin diagram of G.

There is no map from $X^{(\underline{n})}$ to Ran, however, there is one to Ran'; namely it comes from the map

$$X^{\underline{n}} := \prod_i X^{n_i} \to \mathrm{Ran},$$

which is invariant with respect to

$$\Sigma_{\underline{n}} := \prod_i \Sigma_{n_i},$$

where Σ_n is the symmetric group on n letters.

9.3.9. Note that the stack $(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}}$ that appears in (9.10) is canonically isomorphic to

$$\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M}{\times} \mathrm{Zast},$$

where:

- $N(M)$ is the maximal unipotent subgroup of the Levi M;
- The map $\mathrm{Zast} \to \mathrm{Bun}_M$ is the composition

$$\mathrm{Zast} \overset{\mathfrak{s}}{\to} \mathrm{Hecke}(M)_{\mathrm{Ran}'} \overset{\overleftarrow{h}}{\to} \mathrm{Bun}_M .$$

Under this identification, the map

$$(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}} \overset{'\mathsf{p}^{\mathrm{tr}}}{\longrightarrow} \widetilde{\mathrm{Bun}}_{P^-} \overset{\widetilde{\mathsf{q}}^-}{\to} \mathrm{Bun}_M$$

corresponds to

$$\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M}{\times} \mathrm{Zast} \overset{\mathrm{id}\times\mathfrak{s}}{\longrightarrow}$$
$$\to \mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'} \to \mathrm{Hecke}(M)_{\mathrm{Ran}'} \overset{\overrightarrow{h}}{\to} \mathrm{Bun}_M .$$

9.3.10. Let us denote by

$$\Omega_\chi \in \mathrm{Shv}\Big(\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'}\Big)$$

the object equal to the *-direct image along

$$(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}} \simeq \mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M}{\times} \mathrm{Zast} \overset{\mathrm{id}\times\mathfrak{s}}{\longrightarrow}$$
$$\to \mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'}$$

of

$$\Big(('\mathsf{p}^{\mathrm{tr}})^! \circ j_!(\mathsf{e}_{\mathrm{Bun}_{P^-}})\Big) \overset{!}{\otimes} \Big(('\widetilde{\mathsf{p}}^{-,\mathrm{tr}})^! \circ \chi^!(\exp_\omega)\Big) \in \mathrm{Shv}\Big((\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathrm{tr}}\Big).$$

Let us denote by r_2 the projection

$$\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'} \to \mathrm{Hecke}(M)_{\mathrm{Ran}'}.$$

Applying the projection formula, we obtain that the expression in (9.10) identifies with

$$\mathrm{C}^\cdot\left(\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'}, (r_2^! \circ \overrightarrow{h}^!(-)) \overset{!}{\otimes} \Omega_\chi\right) [2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel}]. \tag{9.11}$$

9.3.11. Consider the map

$$\text{transl}_{\rho_P(\omega_X)} : \text{Bun}_{N(M),\rho_M(\omega_X)} \underset{\text{Bun}_M, \overleftarrow{h}}{\times} \text{Hecke}(M)_{\text{Ran}'} \to \to \text{Bun}_{N(M),\rho(\omega_X)} \underset{\text{Bun}_M, \overleftarrow{h}}{\times} \text{Hecke}(M)_{\text{Ran}'}, \tag{9.12}$$

given by (central) translation by $\rho_P(\omega_X)$.

Let r_1 denote the projection

$$\text{Bun}_{N(M),\rho_M(\omega_X)} \underset{\text{Bun}_M, \overleftarrow{h}}{\times} \text{Hecke}(M)_{\text{Ran}'} \to \text{Bun}_{N(M),\rho_M(\omega_X)}.$$

Here is the key computational input in the proof of Theorem 1.4.6:

Proposition 9.3.12. *The pullback of* Ω_χ *along* (9.12)*, viewed as an object of*

$$\text{Shv}(\text{Bun}_{N(M),\rho_M(\omega_X)} \underset{\text{Bun}_M, \overleftarrow{h}}{\times} \text{Hecke}(M)_{\text{Ran}'}),$$

shifted cohomologically by

$$[2\dim(\text{Bun}_M) + \dim.\text{rel} + \delta_{(N_P^-)_{\rho_P(\omega_X)}}],$$

identifies canonically with

$$(r_1^! \circ \chi_M^!(\exp_\omega)) \overset{!}{\otimes} (\text{Sat}_M^{\text{nv}} \circ \text{Fact}(\Omega^{\text{loc}})),$$

where:

- $\chi_M : \text{Bun}_{N(M),\rho_M(\omega_X)} \to \mathbb{G}_a$ *is the counterpart of the map* χ *for* M*.*
- $\text{Sat}_M^{\text{nv}} : \text{Rep}(\check{M}) \to \text{Sph}_M$ *is the naive geometric Satake functor;*
- *By a slight abuse of notation, we denote by* $\text{Sat}_M^{\text{nv}} \circ \text{Fact}(\Omega^{\text{loc}}))$ *the image of the corresponding object under the equivalence*

$$\text{Shv}(\text{Hecke}(M)_{\text{Ran}}) \simeq \text{Shv}(\text{Hecke}(M)_{\text{Ran}'}).$$

Remark 9.3.13. Recall that in order to deduce Theorem 1.4.6 from Theorem 9.2.7 we also need to know that the isomorphism (9.5) is compatible with Ω^{glob}-actions.

This compatibility follows from the corresponding property of the isomorphism of Proposition 9.3.12.

9.3.14. Let us assume Proposition 9.3.12 for a moment and finish the proof of Theorem 9.2.7.

Indeed, combining (9.11) with Proposition 9.3.12, we obtain that the functor

$$\text{coeff}_G^{\text{Vac}} \circ \text{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{(N_P^-)_{\rho_P(\omega_X)}}]$$

is isomorphic to

$$\text{C}^\cdot\left(\text{Bun}_{N(M),\rho_M(\omega_X)} \underset{\text{Bun}_M, \overleftarrow{h}}{\times} \text{Hecke}(M)_{\text{Ran}'}, (r_2^! \circ \overrightarrow{h}^!(-)) \overset{!}{\otimes} (r_1^! \circ \chi_M^!(\exp_\omega)) \overset{!}{\otimes} (\text{Sat}_M^{\text{nv}} \circ \text{Fact}(\Omega^{\text{loc}}))\right)$$

By base change and projection formula, the letter expression identifies with

$$\text{C}^\cdot\left(\text{Bun}_{N(M),\rho_M(\omega_X)}, \mathsf{p}_M^!\left(\overleftarrow{h}_*(\overrightarrow{h}^!(-) \overset{!}{\otimes} (\text{Sat}_M^{\text{nv}} \circ \text{Fact}(\Omega^{\text{loc}})))\right) \overset{!}{\otimes} \chi_M^!(\exp_\omega)\right), \tag{9.13}$$

where

$$\mathsf{p}_M : \text{Bun}_{N(M),\rho_M(\omega_X)} \to \text{Bun}_M.$$

We have, by definition:

$$\overleftarrow{h}_*\left(\overrightarrow{h}^!(-) \overset{!}{\otimes} (\text{Sat}_M^{\text{nv}} \circ \text{Fact}(\Omega))\right) \simeq \text{Fact}(\Omega^{\text{loc}}) \star (-),$$

as endofunctors of $\text{Shv}(\text{Bun}_G)$.

Thus, the expression in (9.13) identifies with

$$\mathrm{C}^{\cdot}\left(\mathrm{Bun}_{N(M),\rho_M(\omega_X)}, \mathsf{p}_M^! \left(\mathrm{Fact}(\Omega^{\mathrm{loc}}) \star (-)\right) \overset{!}{\otimes} \chi_M^!(\exp_\omega)\right) \simeq \mathrm{coeff}_M^{\mathrm{Vac}}\left(\mathrm{Fact}(\Omega^{\mathrm{loc}}) \star (-)\right),$$

as required.

□[Theorem 9.2.7]

9.4. **Proof of Proposition 9.3.12.**

9.4.1. Note that the map χ is a sum of two maps χ^M and χ^{N_P}, where the former is the sum over the simple roots in the Dynkin diagram of M and the latter is the sum over the other simple roots.

Note that the composition

$$\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M}{\times} \mathrm{Zast} \simeq \mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathrm{Bun}}_{P^-} \overset{'\widetilde{\mathsf{p}}^{-,\mathrm{tr}}}{\longrightarrow} \mathrm{Bun}_{N,\rho(\omega_X)} \overset{\chi^M}{\to} \mathbb{G}_a$$

identifies with

$$\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M}{\times} \mathrm{Zast} \overset{r_1}{\to} \mathrm{Bun}_{N(M),\rho(\omega_X)} \overset{\chi_M}{\to} \mathbb{G}_a.$$

Denote by $\Omega_{\chi^{N_P}}$ the object of $\mathrm{Shv}\Big(\mathrm{Bun}_{N(M),\rho(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'}\Big)$ defined in the same way as Ω_χ, but with χ replaced by χ_{N^P}.

By base change Proposition 9.3.12 is equivalent to the isomorphism

$$(\mathrm{transl}_{\rho_P(\omega_X)})^!(\Omega_{\chi^{N_P}})[2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel} + \delta_{(N_P^-)_{\rho_P(\omega_X)}}] \simeq \mathrm{Sat}_M^{\mathrm{nv}} \circ \mathrm{Fact}(\Omega^{\mathrm{loc}}) \tag{9.14}$$

as objects in

$$\mathrm{Shv}(\mathrm{Bun}_{N(M),\rho_M(\omega_X)} \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Ran}'}).$$

9.4.2. Let

$$\overset{\circ}{\mathrm{Zast}} \overset{j_{\mathrm{Zast}}}{\hookrightarrow} \mathrm{Zast}$$

be the open Zastava, i.e., the corresponding open

$$(\mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-})^{\mathrm{tr}} \subset \mathrm{Bun}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}.$$

By a slight abuse of notation, let us denote by $'\mathsf{p}^{\mathrm{tr}}$ the map $\mathrm{Zast} \to \widetilde{\mathrm{Bun}}_{P^-}$ and by $'\overset{\circ}{\mathsf{p}}{}^{\mathrm{tr}}$ the map

$$\overset{\circ}{\mathrm{Zast}} \to \mathrm{Bun}_{P^-}.$$

Since the stacks below involved are smooth, we have

$$('\overset{\circ}{\mathsf{p}}{}^{\mathrm{tr}})^!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}) \simeq \underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}}[2(\dim(\overset{\circ}{\mathrm{Zast}}) - \dim(\mathrm{Bun}_{P^-}))].$$

From here we obtain a map

$$(j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}})[2(\dim(\overset{\circ}{\mathrm{Zast}}) - \dim(\mathrm{Bun}_{P^-}))] \to ('\mathsf{p}^{\mathrm{tr}})^! \circ j_!(\underline{\mathsf{e}}_{\mathrm{Bun}_{P^-}}). \tag{9.15}$$

The next assertion is [Lin, Lemma 4.1.10]:

Lemma 9.4.3. *The map* (9.15) *is an isomorphism.*

9.4.4. From Lemma 9.4.3, we obtain that the object

$$\Omega_{\chi^{N_P}}[2\dim(\mathrm{Bun}_M) + \dim.\mathrm{rel} + \delta_{(N_P^-)_{\rho_P(\omega_X)}}]$$

is isomorphic to

$$(\mathrm{id}\times\mathfrak{s})_*\left(\left(r_2^! \circ (j_{\mathrm{Zast}})_!(\underline{\mathsf{e}}_{\overset{\circ}{\mathrm{Zast}}})\right) \overset{!}{\otimes} \left((\widetilde{\mathsf{p}}^{-,\mathrm{tr}})^! \circ (\chi^{N_P})^!(\exp_\omega)\right)\right)[\dim(\overset{\circ}{\mathrm{Zast}})]. \tag{9.16}$$

The required isomorphism between the pullback of (9.16) along $\mathrm{transl}_{\rho_P(\omega_X)}$ and the object $\mathrm{Sat}_M^{\mathrm{nv}} \circ \mathrm{Fact}(\Omega^{\mathrm{loc}})$ follows from [Ra1, Sect. 4.6.1] (for $P = B$) and [FH, Theoren 1.4.3.1] (for an arbitrary parabolic).

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