SMOOTHNESS OF COHOMOLOGY SHEAVES OF STACKS OF SHTUKAS

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To Gérard Laumon, with deepest admiration and gratitude.

Abstract. We prove, for all reductive groups, that the cohomology sheaves with compact support of stacks of shtukas are ind-lisse over $(X \smallsetminus N)^I$ and that their geometric generic fibers are equipped with an action of $\mathrm{Weil}(X \smallsetminus N, \overline{\eta})^I$. Our method does not use any compactification of stacks of shtukas.

Contents

Introduction

Let X be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$. We denote by F its function field. Let $N \subset X$ be a finite subscheme.

Let G be a connected reductive group over F. Let ℓ be a prime number not dividing q. Let E be a finite extension of $\mathbb{Q}_\ell$ containing a square root of q, with ring of integers $\mathcal{O}_E$ and residual field k_E. Let $\Lambda \in \{E, \mathcal{O}_E, k_E\}$.

Until the last section, we assume that G is split to simplify the notations. Let $\widehat{G}$ be the Langlands dual group of G over Λ. We fix a lattice Ξ in $Z_G(F)\backslash Z_G(\mathbb{A})$ as in [Laf18], where Z_G is the center of G (when G is semisimple, we can take $\Xi = 1$).

The cohomology sheaves of stacks of shtukas are defined in [Laf18, Section 4] for $\Lambda = E$ and in [Laf18, Section 13] [Xue20c, Section 1] for $\Lambda = \mathcal{O}_E$. For $\Lambda = k_E$ the method is the same. We refer to [Xue22] for a detailed reminder (for $\Lambda = E$). Here is a brief reminder:

Let I be a finite set. We have the stack classifying G-shtukas with I-legs and level N:

$$\mathfrak{p} : \mathrm{Cht}_{G,N,I} \to (X \smallsetminus N)^I$$

By the geometric Satake equivalence with coefficients in Λ ([MV07, Theorem 14.1], [Gai07, Theorem 2.2, Theorem 2.6], [Ric14], [CvdHS23], the properties that we needed are stated in [Laf18, Theoreme 1.17]), for any W a finite type Λ-linear representation of $\widehat{G}^I$, we have a canonical perverse sheaf $\mathcal{F}_{G,N,I,W}$ on $\mathrm{Cht}_{G,N,I}$ for the perverse normalization relative to $(X \smallsetminus N)^I$. Its support (denoted by $\mathrm{Cht}_{G,N,I,W}$) is a Deligne-Mumford stack locally of finite type. When W is irreducible, $\mathcal{F}_{G,N,I,W}$ is (not canonically) isomorphic to the intersection complex of $\mathrm{Cht}_{G,N,I,W}$.

To define the cohomology we need stacks of finite type. We have the Harder-Narasimhan truncations indexed by $\widehat{\Lambda}^+_{G^{\mathrm{ad}}}$, the set of dominant coweights of G^{ad} (the adjoint group of G). For every $\mu \in \widehat{\Lambda}^+_{G^{\mathrm{ad}}}$ we have a truncated open substack of shtukas $\mathrm{Cht}^{\leq\mu}_{G,N,I,W}$ in $\mathrm{Cht}_{G,N,I,W}$. The quotient $\mathrm{Cht}^{\leq\mu}_{G,N,I,W}/\Xi$ is of finite type (this is the reason why we need to consider the truncation by μ and the quotient by Ξ). We denote by

$$\mathfrak{p}^{\leq\mu} : \mathrm{Cht}^{\leq\mu}_{G,N,I}/\Xi \to (X \smallsetminus N)^I$$

We define the complex of truncated cohomology sheaves:

$$\mathcal{H}^{\leq\mu}_{G,N,I,W} := (\mathfrak{p}^{\leq\mu})_! \mathcal{F}_{G,N,I,W}$$

For $\Lambda = E$, this complex lives in $D^b_c((X \smallsetminus N)^I, \Lambda)$. It is bounded in degrees $[-d, d]$ where $d = \dim \mathrm{Cht}_{G,N,I,W} - \dim X^I$. For $\Lambda = \mathcal{O}_E$ or k_E, this complex lives in $D^-_c((X \smallsetminus N)^I, \Lambda)$. (See Remark 0.0.3.)

For any $j \in \mathbb{Z}$, we have the degree j truncated cohomology sheaf with compact support :

$$\mathcal{H}^{j,\leq\mu}_{G,N,I,W} := R^j(\mathfrak{p}^{\leq\mu})_! \mathcal{F}_{G,N,I,W}.$$

It is a constructible Λ-sheaf over $(X \smallsetminus N)^I$.

We define the complex of cohomology sheaves and the degree j cohomology sheaf as the following inductive limits:

$$\mathcal{H}_{G,N,I,W} := \varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{G,N,I,W};$$

$$\mathcal{H}^j_{G,N,I,W} := \varinjlim_{\mu} \mathcal{H}^{j,\leq\mu}_{G,N,I,W}.$$

The cohomology sheaf $\mathcal{H}^j_{G,N,I,W}$ lives in the category of abstract inductive limits of constructible Λ-sheaves over $(X \smallsetminus N)^I$. The complex $\mathcal{H}_{G,N,I,W}$ lives in the derived category of abstract inductive limits of complexes of constructible Λ-sheaves over $(X \smallsetminus N)^I$ (also known as ind-completion).

When I is the empty set and W the trivial representation, $\mathcal{H}^0_{G,N,I,W}$ is the vector space of automorphic forms with level N. For general I and W, an important property of $\mathcal{H}_{G,N,I,W}$ is that it is equipped with an action of the Hecke algebra and an action of the partial Frobenius morphisms. Note that these actions do not preserve $\mathcal{H}^{\leq\mu}_{G,N,I,W}$.

If the morphism from $\mathrm{Cht}_{G,N,I,W}/\Xi$ to $(X \smallsetminus N)^I$ is proper [1], we know that the cohomology sheaves are lisse over $(X \smallsetminus N)^I$. In general this morphism is not proper (not even of finite type), the question is whether the cohomology sheaves are still lisse over $(X \smallsetminus N)^I$. Our main result gives a positive answer to this question:

Theorem 0.0.1. *(Theorem 3.2.3, Theorem 3.3.1) For any $j \in \mathbb{Z}$, for any μ sufficiently regular (i.e. far away from every wall in the Weyl chamber), the constructible Λ-sheaf $\mathcal{H}^{j,\leq\mu}_{G,N,I,W}$ is lisse over $(X \smallsetminus N)^I$. The ind-constructible Λ-sheaf $\mathcal{H}^{j}_{G,N,I,W}$ is ind-lisse over $(X \smallsetminus N)^I$.*

Here ind-lisse means inductive limit of lisse sheaves. The proof of Theorem 0.0.1 uses a "Zorro lemma" argument and the following intermediate result Proposition 0.0.2.

Let η be the generic point of X and $\overline{\eta}$ a geometric point over η. Let η_I be the generic point of X^I and $\overline{\eta_I}$ a geometric point over η_I. We refer to 1.1.6 for more details. Let $\mathrm{Weil}(\eta, \overline{\eta})$ be the Weil group of $\pi_1(\eta, \overline{\eta})$.

Proposition 0.0.2. *(Proposition 2.2.1) The geometric generic fiber $\mathcal{H}^{j}_{G,N,I,W}\big|_{\overline{\eta_I}}$ is equipped with a canonical action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.*

The proof of Proposition 0.0.2 uses Drinfeld's lemma and some finiteness property of $\mathcal{H}^{j}_{G,N,I,W}\big|_{\overline{\eta_I}}$.

Remark 0.0.3. *We mentioned above that for $\Lambda = \mathcal{O}_E$ or k_E, the complex $\mathcal{H}^{\leq\mu}_{G,N,I,W}$ lives only in $D^-_c((X \smallsetminus N)^I, \Lambda)$. To see this, for example consider the stacks of shtukas without leg and without level, which is the discrete stack $\mathrm{Bun}_G(\mathbb{F}_q)$. This stack contains $[\cdot/G(\mathbb{F}_q)]$ (corresponding to the trivial G-bundle in $\mathrm{Bun}_G(\mathbb{F}_q)$). When ℓ divides the cardinality of $G(\mathbb{F}_q)$, the complex $H^*_c([\cdot/G(\mathbb{F}_q)], \mathbb{Z}_\ell)$ is unbounded below. In this case, $\mathcal{H}^{\leq\mu}_{G,N,I,W}\Big|_{\overline{\eta_I}}$ is not a perfect complex. Even when we suppose that ℓ does not divide the cardinal of $G(\mathbb{F}_q)$, we do not know if $\mathcal{H}^{\leq\mu}_{G,N,I,W}\Big|_{\overline{\eta_I}}$ is a perfect complex. So we do not know if $\mathcal{H}^{\leq\mu}_{G,N,I,W}$ is in $D_{\mathrm{cons}}((X \smallsetminus N)^I, \Lambda)$ in the sense of proetale topos of Bhatt and Scholze. At the end, we do not know if $\mathcal{H}_{G,N,I,W}$ is in $D_{\mathrm{indlisse}}((X \smallsetminus N)^I, \Lambda)$.*

For $\Lambda = E$, the complex $\mathcal{H}^{\leq\mu}_{G,N,I,W}$ lives in $D^b_c((X \smallsetminus N)^I, \Lambda)$. There is no such problem.

Remark 0.0.4. *Even in the case where $\mathcal{H}_{G,N,I,W}$ is in $D_{\mathrm{indlisse}}((X \smallsetminus N)^I)$, I do not know how to prove that $\mathcal{H}_{G,N,I,W}$ lives in $(D_{\mathrm{indlisse}}(X \smallsetminus N))^{\otimes I}$.*

Relation with literature. The result of smoothness is used in [AGKRRV].

The results of this article (and the "Zorro lemma" argument) are generalized in [Sal23], [Ete23] and [EX24].

[1]For G anisotropic, for example a division algebra, see [Lau07, Theorem A] for the condition when this morphism if proper. However, when G is split but not a torus, this morphism is never of finite type.

Notations and conventions. To simplify the notations, we will write $\mathcal{H}_{I,W}$ instead of the degree j cohomology sheaf $\mathcal{H}^j_{G,N,I,W}$, except in Section 3.3.

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1. Reminder on Drinfeld's lemmas

1.1. Sheaves with an action of the partial Frobenius morphisms.

1.1.1. Let I be a finite set. For any $i \in I$, let

$$\mathrm{Frob}_{\{i\}} : X^I \to X^I \tag{1.1}$$

be the morphism sending $(x_j)_{j\in I}$ to $(x'_j)_{j\in I}$, with $x'_i = \mathrm{Frob}(x_i)$ and $x'_j = x_j$ if $j \neq i$, where Frob is the absolute Frobenius morphism of X (i.e. identity on the topological space and q-th power on the structure sheaf). They commute with each other and the product $\prod_{i\in I} \mathrm{Frob}_{\{i\}}$ is the total Frobenius morphism Frob, i.e. the absolute Frobenius morphism of X^I.

1.1.2. Let $\mathcal{G}$ be a sheaf over X^I. We say that $\mathcal{G}$ is *equipped with an action of the partial Frobenius morphisms* if there exist isomorphisms of sheaves over X^I, defined for every $i \in I$:

$$F_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{G} \xrightarrow{\sim} \mathcal{G}, \tag{1.2}$$

that commute with one another, such that the composition for all $i \in I$ is the total Frobenius isomorphism $F : \mathrm{Frob}^* \mathcal{G} \xrightarrow{\sim} \mathcal{G}$.

Example 1.1.3. Let $\mathcal{G}$ be a sheaf over X^I of the form $\mathcal{G} = \boxtimes_{i\in I}\mathcal{F}_i$, where every $\mathcal{F}_i$ is a sheaf over X. Then $\mathcal{G}$ is equipped with an action of the partial Frobenius morphisms.

Example 1.1.4. Let $X = \mathbb{P}^1$. The pullback of the Artin-Schreier sheaf on $\mathbb{A}^1$ by the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, extended by zero from $\mathbb{A}^1 \times \mathbb{A}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$, gives a sheaf on X^2 which cannot have an action of the partial Frobenius morphisms.

1.1.5. We fix a geometric point $\overline{\eta} = \mathrm{Spec}\, \overline{F}$ over the generic point $\eta = \mathrm{Spec}\, F$ of X. We denote by

$$(\eta)^I := \eta \times_{\mathrm{Spec}\, \mathbb{F}_q} \cdots \times_{\mathrm{Spec}\, \mathbb{F}_q} \eta \quad \text{and} \quad (\overline{\eta})^I := \overline{\eta} \times_{\mathrm{Spec}\, \overline{\mathbb{F}_q}} \cdots \times_{\mathrm{Spec}\, \overline{\mathbb{F}_q}} \overline{\eta}.$$

Note that $(\eta)^I$ and $(\overline{\eta})^I$ are integral schemes.

1.1.6. We fix a geometric point $\overline{\eta_I} = \mathrm{Spec}\, \overline{F_I}$ over the generic point $\eta_I = \mathrm{Spec}\, F_I$ of X^I. We fix a specialization map in X^I

$$\mathfrak{sp} : \overline{\eta_I} \to \Delta(\overline{\eta})$$

where $\Delta : X \to X^I$ is the diagonal inclusion. The specialization map $\mathfrak{sp}$ induces a morphism $\overline{\eta_I} \to (\overline{\eta})^I$.

1.1.7. As in [Laf18, Remarque 8.18], we define

$$\mathrm{FWeil}(\eta_I, \overline{\eta_I}) := \{\varepsilon \in \mathrm{Aut}_{\overline{\mathbb{F}_q}}(\overline{F_I}) \mid \exists\, (n_i)_{i\in I} \in \mathbb{Z}^I, \varepsilon\big|_{(F_I)^{\mathrm{perf}}} = \prod_{i\in I} (\mathrm{Frob}_{\{i\}})^{n_i}\}.$$

where $(F_I)^{\mathrm{perf}}$ is the perfection of F_I and $\mathrm{Frob}_{\{i\}}$ is the partial Frobenius morphism defined in 1.1.1. We have a commutative diagram where the lines are exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I}) & \longrightarrow & \mathrm{FWeil}(\eta_I, \overline{\eta_I}) & \longrightarrow & \mathbb{Z}^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \Psi & & \downarrow \simeq & & \\ 0 & \longrightarrow & \pi_1^{\mathrm{geom}}(\eta, \overline{\eta})^I & \longrightarrow & \mathrm{Weil}(\eta, \overline{\eta})^I & \longrightarrow & \mathbb{Z}^I & \longrightarrow & 0 \end{array} \tag{1.3}$$

where morphism Ψ is given by sending ε to $(\mathrm{Frob}_{\{i\}}^{-n_i} \circ \varepsilon_i)_{i\in I}$, where each ε_i is the restriction of ε to $\overline{F}$ via $\overline{\eta_I} \to (\overline{\eta})^I \xrightarrow{\mathrm{pr}_i} \overline{\eta}$.

1.1.8. Let $\mathcal{G}$ be a sheaf over η_I, equipped with an action of the partial Frobenius morphisms. Then $\mathcal{G}\big|_{\overline{\eta_I}}$ is equipped with a canonical action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ in the following way:

for any $\varepsilon \in \mathrm{FWeil}(\eta_I, \overline{\eta_I})$ with $\varepsilon\big|_{(F_I)^{\mathrm{perf}}} = \prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i}$, it induces a commutative diagram (which is not Cartesian):

$$\begin{array}{ccc} \mathrm{Spec}\, \overline{F_I} & \xrightarrow[\simeq]{\mathrm{Spec}\, \varepsilon} & \mathrm{Spec}\, \overline{F_I} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F_I)^{\mathrm{perf}} & \xrightarrow[\simeq]{\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i}} & \mathrm{Spec}(F_I)^{\mathrm{perf}} \end{array}$$

We make it into a Cartesian diagram:

$$\begin{array}{cccc} \mathrm{Spec}\, \overline{F_I} & & & \\ & \searrow^{\simeq} & \overset{\mathrm{Spec}\,\varepsilon}{\underset{\simeq}{\longrightarrow}} & \\ & (\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i})\, \mathrm{Spec}\, \overline{F_I} & \xrightarrow[\simeq]{} & \mathrm{Spec}\, \overline{F_I} \\ & \downarrow & & \downarrow \\ & \mathrm{Spec}(F_I)^{\mathrm{perf}} & \xrightarrow[\simeq]{\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i}} & \mathrm{Spec}(F_I)^{\mathrm{perf}} \end{array}$$

We deduce an isomorphism of schemes over $\mathrm{Spec}(F_I)^{\mathrm{perf}}$:

$$(\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I}) \xrightarrow{\sim} \overline{\eta_I}.$$

In particular, it is a specialization map in $\mathrm{Spec}(F_I)^{\mathrm{perf}}$. We denote it by $\mathfrak{sp}_\varepsilon$.

The action of ε on $\mathcal{F}\big|_{\overline{\eta_I}}$ is defined to be the composition:

$$\mathcal{F}\big|_{\overline{\eta_I}} \xrightarrow{\mathfrak{sp}_\varepsilon^*} \mathcal{F}\big|_{(\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})} = \Big((\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i})^* \mathcal{F}\Big)\Big|_{\overline{\eta_I}} \xrightarrow{\prod_{i\in I} F_{\{i\}}^{n_i}} \mathcal{F}\big|_{\overline{\eta_I}} \tag{1.4}$$

where the last morphism is the product of the partial Frobenius morphisms (which are isomorphisms, and over $\mathrm{Spec}(F_I)^{\mathrm{perf}}$ the inverses $\mathrm{Frob}_{\{i\}}^{-1}$ are well defined).

1.2. **Drinfeld's lemma.**

1.2.1. An action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ on a finite type Λ-module is said to be continuous if the action of $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ is continuous.

More generally, an action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ on a Λ-module M is said to be continuous if M is an inductive limit of Λ-submodules of finite type which are stable under $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ and on which the action of $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ is continuous.

Lemma 1.2.2. *(Drinfeld's lemma) A continuous action of* $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ *on a* Λ*-module of finite type factors through* $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Proof. For $\Lambda = \mathcal{O}_E$ or k_E, it is proved in [Dri87, Proposition 1.1], [Dri89, Proposition 6.1] and recalled in [Laf18, Lemme 8.2]. For $\Lambda = E$, it is proved by Drinfeld (unpublished) and recalled in [Xue20b, Lemma 3.2.10].

Let's briefly recall how we deduce the case $\Lambda = E$ from the case $\Lambda = k_E$. We defined the morphism $\Psi : \mathrm{FWeil}(\eta_I, \overline{\eta_I}) \to \mathrm{Weil}(\eta, \overline{\eta})^I$ in 1.1.7. By [Dri89, Proposition 6.1], $\mathrm{Ker}(\Psi)$ is equal to the intersection of all open subgroups of $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ which are normal in $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$. Let $\rho : \mathrm{FWeil}(\eta_I, \overline{\eta_I}) \to GL_r(E)$ be a continuous morphism, then $\mathrm{Ker}(\rho|_{\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})})$ is normal in $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ and closed in $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$, and $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I}) / \mathrm{Ker}(\rho|_{\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})})$ is topologically finitely generated (i.e. there exists a dense finitely generated subgroup). Using properties of profinite groups, we can prove that such a closed subgroup of $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ contains $\mathrm{Ker}(\Psi)$. □

Lemma 1.2.3. *Let A be a finitely generated commutative Λ-algebra. A continuous A-linear action of* $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ *on an A-module of finite type factors through* $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Proof. For $\Lambda = E$, it is [Xue20b, Lemma 3.2.13]. For $\Lambda = \mathcal{O}_E$, it is [Xue20c, Lemma 8.2.4]. For $\Lambda = k_E$ the proof is similar. □

1.3. **Toy model: constructible sheaves.**

Let's recall a simple but important lemma:

Lemma 1.3.1. *(*[Lau04, Lemma 9.2.1]*, recalled in* [Laf18, Lemme 8.12]*) Let Z be a proper closed subscheme of X^I, stable under the action of the partial Frobenius morphisms, then Z is included in a finite union of vertical divisors of X^I (a vertical divisor is the inverse image of a closed point by one of the projections $X^I \to X$).*

For a Λ-constructible sheaf, since it is lisse over an open subscheme, we have the following property:

Lemma 1.3.2. *(consequence of* [Lau04] *Lemma 9.2.1) Let $\mathcal{G}$ be a Λ-constructible sheaf over X^I, equipped with an action of the partial Frobenius morphisms. Then there exists an open dense subscheme U of X such that $\mathcal{G}$ is lisse over U^I.*

Proof. Let Ω be the largest open subscheme of X^I such that $\mathcal{G}$ is lisse over Ω. Then the complement of Ω is a proper closed subscheme of X^I, stable under the action of the partial Frobenius morphisms. By Lemma 1.3.1, this closed subscheme is included in a finite union of vertical divisors of X^I. So there exists a dense open subscheme U of X such that $U^I \subset \Omega$. □

1.3.3. Let $\mathcal{G}$ be a Λ-constructible sheaf over X^I, equipped with an action of the partial Frobenius morphisms.

(1) On the one hand, $\mathcal{G}\big|_{\overline{\eta_I}}$ is equipped with a continuous action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$. By Lemma 1.2.2, this action factors through $\mathrm{Weil}(\eta, \overline{\eta})^I$.

(2) On the other hand, by Lemma 1.3.2, $\mathcal{G}$ is lisse over U^I. In particular, $\mathcal{G}$ is lisse over $(\overline{\eta})^I$.

By Lemma 1.3.4 below, $\mathcal{G}\big|_{(\overline{\eta})^I}$ is a constant sheaf over $(\overline{\eta})^I$.

Lemma 1.3.4. *Let $\mathcal{G}$ be an ind-lisse Λ-sheaf over $(\overline{\eta})^I$ equipped with an action of* $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$. *If the action factors through* $\mathrm{Weil}(\eta, \overline{\eta})^I$, *then $\mathcal{G}$ is a constant sheaf over $(\overline{\eta})^I$.*

Proof. By hypothesis, the action of $\mathrm{Ker}\,\Psi = \mathrm{Ker}(\mathrm{FWeil}(\eta_I, \overline{\eta_I}) \to \mathrm{Weil}(\eta, \overline{\eta})^I)$ on $\mathcal{G}\big|_{\overline{\eta_I}}$ is trivial. Note that by (1.3), we have $\mathrm{Ker}\,\Psi \cong \mathrm{Ker}(\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I}) \to \pi_1^{\mathrm{geom}}(\eta, \overline{\eta})^I)$. Now let δ be the generic point of $(\overline{\eta})^I$. We have the following commutative diagram

$$\begin{tikzcd}
\overline{\eta_I} \arrow[r] \arrow[dr] \arrow[ddr] & \delta \arrow[r, "\text{generic}"] \arrow[d] & (\overline{\eta})^I \arrow[d] & & \\
& (\eta_I)_{\overline{\mathbb{F}_q}} \arrow[r, "\text{generic}"] \arrow[d] & (\eta)^I_{\overline{\mathbb{F}_q}} \arrow[r] \arrow[d] & X^I_{\overline{\mathbb{F}_q}} \arrow[r] \arrow[d] & \mathrm{Spec}\,\overline{\mathbb{F}_q} \arrow[d] \\
& \eta_I \arrow[r, "\text{generic}"] & (\eta)^I \arrow[r] & X^I \arrow[r] & \mathrm{Spec}\,\mathbb{F}_q
\end{tikzcd}$$

Thus we have a canonical morphism $\pi_1(\delta, \overline{\eta_I}) \to \pi_1((\eta_I)_{\overline{\mathbb{F}_q}}, \overline{\eta_I}) = \pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$. By definition $\pi_1(\delta, \overline{\eta_I}) \subset \mathrm{Ker}\,\Psi$, we deduce that the action of $\pi_1(\delta, \overline{\eta_I})$ on $\mathcal{G}\big|_{\overline{\eta_I}}$ is trivial.

Since $\mathcal{G}$ is ind-lisse over $(\overline{\eta})^I$, the action of $\pi_1(\delta, \overline{\eta_I})$ on $\mathcal{G}\big|_{\overline{\eta_I}}$ factors through the quotient $\pi_1((\overline{\eta})^I, \overline{\eta_I})$, and this action is also trivial. We deduce that $\mathcal{G}\big|_{(\overline{\eta})^I}$ is a constant sheaf over $(\overline{\eta})^I$. □

1.4. **Difficulty for a general ind-constructible sheaf.**

1.4.1. A constructible Λ-sheaf over a scheme is lisse over an open subscheme. However, an ind-constructible Λ-sheaf over a scheme may not be ind-lisse over any open subscheme. For example, let $I = \{1, 2\}$ and $(\Lambda)_n$ be the extension by zero of the constant sheaf Λ over $\mathrm{Frob}_{\{1\}}^n(\Delta)$, where Δ is the image of the diagonal morphism $X \hookrightarrow X^2$. Let $\mathcal{G}_m = \oplus_{-m \leq n \leq m} (\Lambda)_n$. It is a constructible Λ-sheaf, lisse over the open subscheme $X^2 - \cup_{-m \leq n \leq m} \mathrm{Frob}_{\{1\}}^n(\Delta)$. However, the ind-constructible Λ-sheaf $\mathcal{G} = \varinjlim_m \mathcal{G}_m = \oplus_{n\in\mathbb{Z}}(\Lambda)_n$ is not ind-lisse over any open subscheme.

For this reason, there is no analogue of Lemma 1.3.2 for general ind-constructible Λ-sheaves. Note that in the above example, the sheaf $\mathcal{G}$ is equipped with an action of the partial Frobenius morphisms. But it is not ind-lisse even over $\eta \times_{\mathbb{F}_q} \eta$.

1.4.2. The cohomology sheaf $\mathcal{H}_{I,W}$ is an ind-constructible Λ-sheaf, so we cannot apply directly 1.3.3 to show that it is constant over $(\overline{\eta})^I$. In the next section, instead of 1.3.3 (1), we will prove some finiteness property of $\mathcal{H}_{I,W}$ then apply Lemma 1.2.3. Instead of 1.3.3 (2), we will use another method to prove that $\mathcal{H}_{I,W}$ is lisse over $(\overline{\eta})^I$.

2. Application of Drinfeld's lemma to the cohomology sheaves

2.1. **Finiteness properties of cohomology groups.**

2.1.1. By [Laf18, Sections 3 and 4], the cohomology sheaf $\mathcal{H}_{I,W}$ is equipped with an action of the partial Frobenius morphisms. By 1.1.8, $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is equipped with a canonical action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$. This action is continuous in the sense of 1.2.1, because the action of $\pi_1^{\mathrm{geom}}(\eta_I, \overline{\eta_I})$ on each $\mathcal{H}_{I,W}^{\leq\mu}\Big|_{\overline{\eta_I}}$ is continuous.

2.1.2. Let v be a place of $X \smallsetminus N$. We denote by $\mathcal{O}_v$ the complete local ring at v and F_v its field of fractions. Let $\mathscr{H}_{G,v} := C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \Lambda)$ be the local Hecke algebra of G at the place v. Note that $\mathscr{H}_{G,v}$ is a finitely generated Λ-algebra. By [Laf18, Section 4.4], $\mathcal{H}_{I,W}\big|_{(X\smallsetminus(N\cup v))^I}$ is equipped with a canonical action of $\mathscr{H}_{G,v}$.

2.1.3. We denote by $\mathrm{Rep}_\Lambda(\widehat{G})$ (resp. $\mathrm{Rep}_\Lambda(\widehat{G}^I)$) the category of finite type Λ-linear representations of $\widehat{G}$ (resp. $\widehat{G}^I$). We denote by $\mathrm{Rep}_\Lambda(\widehat{G})^{\mathrm{free}}$ the category of representation of $\widehat{G}$ on a free Λ-module of finite type. Note that for $\Lambda = E$ these categories are semisimple, but for $\Lambda = \mathcal{O}_E$ or k_E they are not semisimple.

Let's recall the Eichler-Shimura relations:

Proposition 2.1.4. *(For $\Lambda = E$, see* [Laf18, Proposition 7.1]*. For $\Lambda = \mathcal{O}_E$, see* [XZ17, Section 6.2] [Xue20c, Proposition 7.2.6]*, the proof for $\Lambda = k_E$ is similar.) For any finite set $I = \widetilde{I} \cup \{0\}$ and $W \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, there exists $M \in \mathrm{Rep}_\Lambda(\widehat{G})^{\mathrm{free}}$, such that*

$$\sum_{\alpha=0}^{\mathrm{rk}\, M}(-1)^\alpha S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v} \circ (F_{\{0\}}^{\deg(v)})^\alpha = 0 \quad \textit{in} \tag{2.1}$$

$$\mathrm{Hom}_{D_c^-((X\smallsetminus N)^{\widetilde{I}}\times v,\Lambda)}(\mathcal{H}_{I,W}^{\leq\mu}\Big|_{(X\smallsetminus N)^{\widetilde{I}}\times v}, \mathcal{H}_{I,W}^{\leq\mu+\kappa}\Big|_{(X\smallsetminus N)^{\widetilde{I}}\times v}).$$

where $S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v}$ are the excursion operators constructed in loc.cit..

Proposition 2.1.5. *(*[Laf18] *Proposition 6.2, the proof works for any coefficients) For any place v of $X \smallsetminus N$ and any $V \in \mathrm{Rep}_\Lambda(\widehat{G})$, the excursion operator $S_{V,v}$ defined in loc.cit., which is a morphism of sheaves over $(X \smallsetminus N)^I$, extends the Hecke operator $T(h_{V,v})$, which is a morphism of sheaves over $(X \smallsetminus (N \cup v))^I$.*

Now we can prove the finiteness property of cohomology sheaves:

Proposition 2.1.6. *The geometric generic fiber of the cohomology sheaf $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is an increasing union of sub Λ-modules $\mathfrak{M}$ which are stable under the action of* $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$*, and for which there exists a family $(v_i)_{i\in I}$ of closed points in $X \smallsetminus N$ (depending on $\mathfrak{M}$) such that $\mathfrak{M}$ is stable under the action of $\otimes_{i\in I}\mathscr{H}_{G,v_i}$ and is of finite type as module over $\otimes_{i\in I}\mathscr{H}_{G,v_i}$.*

The proof is inspired by the proof of [Laf18, Lemme 8.30].

Proof. For every $\mu \in \widehat{\Lambda}^+_{G^{\mathrm{ad}}}$, we choose a dense open subscheme Ω of $(X \smallsetminus N)^I$ such that $\mathcal{H}^{\leq\mu}_{I,W}\big|_{\Omega}$ is lisse. We choose a closed point v of Ω. Let v_i be the image of v under the projection to the i-th factor $X^I \xrightarrow{\mathrm{pr}_i} X$. Then $\times_{i\in I} v_i$ is a finite union of closed points containing v. Let $\mathfrak{M}_\mu$ be the image of

$$\sum_{(n_i)_{i\in I}\in\mathbb{N}^I} (\otimes_{i\in I}\mathscr{H}_{G,v_i}) \cdot \Big(\prod_{i\in I} F^{n_i}_{\{i\}}\big((\prod_{i\in I}\mathrm{Frob}^{n_i}_{\{i\}})^*\mathcal{H}^{\leq\mu}_{I,W}\big)\Big)\Big|_{\overline{\eta_I}} \tag{2.2}$$

in $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$. We have

$$\mathcal{H}_{I,W}\big|_{\overline{\eta_I}} = \bigcup_\mu \mathfrak{M}_\mu. \tag{2.3}$$

By definition, the sub Λ-module $\mathfrak{M}_\mu$ is stable under the action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$.

We only need to prove that $\mathfrak{M}_\mu$ is of finite type as a $\otimes_{i\in I}\mathscr{H}_{G,v_i}$-module. We fix a geometric point $\overline{v}$ over v and a specialization map $\mathfrak{sp}_v : \overline{\eta_I} \to \overline{v}$. For any n_i, since

$$F^{\deg(v_i)n_i}_{\{i\}} : (\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^*\mathcal{H}^{\leq\mu}_{I,W} \to \mathcal{H}_{I,W}$$

is a morphism of sheaves, the specialization map $\mathfrak{sp}_v$ induces a commutative diagram

$$\begin{array}{ccc} (\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^*\mathcal{H}^{\leq\mu}_{I,W}\big|_{\overline{v}} & \xrightarrow[\simeq]{\mathfrak{sp}_v^*} & (\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^*\mathcal{H}^{\leq\mu}_{I,W}\big|_{\overline{\eta_I}} \\ \Big\downarrow F^{\deg(v_i)n_i}_{\{i\}} & & \Big\downarrow F^{\deg(v_i)n_i}_{\{i\}} \\ \mathcal{H}_{I,W}\big|_{\overline{v}} & \xrightarrow{\mathfrak{sp}_v^*} & \mathcal{H}_{I,W}\big|_{\overline{\eta_I}} \end{array} \tag{2.4}$$

Note that $\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}}(v) = v \in \Omega$, thus $v \in (\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^{-1}\Omega$. The sheaf $(\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^*\mathcal{H}^{\leq\mu}_{I,W}$ is lisse over $(\mathrm{Frob}^{\deg(v_i)n_i}_{\{i\}})^{-1}\Omega$. Thus the upper line of (2.4) is an isomorphism.

By the Eichler-Shimura relations given by Proposition 2.1.4, for each $i \in I$, we have

$$\sum_{\alpha=0}^{\mathrm{rk}\, M}(-1)^{\alpha} S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}(F_{\{i\}}^{\deg(v_i)})^{\alpha} = 0 \quad \text{in } \mathrm{Hom}(\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{v}}, \mathcal{H}_{I,W}\big|_{\overline{v}})$$

We deduce that

$$\begin{aligned} & F_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M}\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{v}}\big) \\ \subset & \sum_{\alpha=0}^{\mathrm{rk}\, M-1} S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}F_{\{i\}}^{\deg(v_i)\alpha}\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\alpha})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{v}}\big) \end{aligned}$$

viewed in $\mathcal{H}_{I,W}\big|_{\overline{v}}$. Since $S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}$ and $F_{\{i\}}$ are morphisms of sheaves, they commute with $\mathfrak{sp}_v^*$. We have

$$\begin{aligned} & F_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M}\mathfrak{sp}_v^*\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{v}}\big) \\ \subset & \sum_{\alpha=0}^{\mathrm{rk}\, M-1} S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}F_{\{i\}}^{\deg(v_i)\alpha}\mathfrak{sp}_v^*\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\alpha})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{v}}\big) \end{aligned}$$

viewed in $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$. Since the upper line of (2.4) is an isomorphism, we deduce that

$$\begin{aligned} & F_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M}\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\,\mathrm{rk}\, M})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{\eta_I}}\big) \\ \subset & \sum_{\alpha=0}^{\mathrm{rk}\, M-1} S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}F_{\{i\}}^{\deg(v_i)\alpha}\big((\mathrm{Frob}_{\{i\}}^{\deg(v_i)\alpha})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big|_{\overline{\eta_I}}\big) \end{aligned} \tag{2.5}$$

By Proposition 2.1.5, $S_{\wedge^{\mathrm{rk}\, M-\alpha}M,v_i}$ acts over $\overline{\eta_I}$ by an element of $\mathscr{H}_{G,v_i}$. We deduce that $\mathfrak{M}_\mu$ is equal to the image of

$$\sum_{(n_i)_{i\in I}\in\prod_{i\in I}\{0,1,\cdots,\deg(v_i)(\mathrm{rk}\, M-1)\}} (\otimes_{i\in I}\mathscr{H}_{G,v_i})\cdot\Big(\prod_{i\in I}F_{\{i\}}^{n_i}\big((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})^{*}\mathcal{H}_{I,W}^{\leq\mu}\big)\Big)\Big|_{\overline{\eta_I}} \tag{2.6}$$

in $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$. Thus $\mathfrak{M}_\mu$ is of finite type as a $\otimes_{i\in I}\mathscr{H}_{G,v_i}$-module. □

Remark 2.1.7. *In* [Xue20b], *we proved a stronger result: $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is of finite type as a module over a local Hecke algebra. (In Proposition 2.1.6 above $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is only an inductive limit of such modules $\mathfrak{M}$, and the local Hecke algebra changes with each $\mathfrak{M}$.) The proof uses the constant term morphisms of the cohomology of stacks of shtukas (for the moment only written for split groups) and doesn't use the Eichler-Shimura relations.*

The advantage of the proof of Proposition 2.1.6 given here is that it is easily generalized to not necessarily split groups in Section 4.

2.2. Action of the Weil group.

Proposition 2.2.1. *The action of* $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ *on* $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ *factors through* $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Proof. By Proposition 2.1.6, $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is an increasing union of sub $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$-representations $\mathfrak{M}$. For every $\mathfrak{M}$, we apply Lemma 1.2.3 to $A = \otimes_{i\in I}\mathscr{H}_{G,v_i}$ and $M = \mathfrak{M}$. We deduce that the action of $\mathrm{FWeil}(\eta_I, \overline{\eta_I})$ on $\mathfrak{M}$ factors through $\mathrm{Weil}(\eta, \overline{\eta})^I$. □

2.3. Constancy over $(\overline{\eta})^I$.

Proposition 2.3.1. *The ind-constructible sheaf* $\mathcal{H}_{I,W}\big|_{(\overline{\eta})^I}$ *is ind-lisse over* $(\overline{\eta})^I$.

By Lemma A.0.3, it is enough to prove that for any geometric point $\overline{x}$ of $(\overline{\eta})^I$ and any specialization map

$$\mathfrak{sp}_{\overline{x}} : \overline{\eta_I} \to \overline{x}$$

the induced morphism

$$\mathfrak{sp}_{\overline{x}}^* : \mathcal{H}_{I,W}\big|_{\overline{x}} \to \mathcal{H}_{I,W}\big|_{\overline{\eta_I}} \tag{2.7}$$

is an isomorphism. The injectivity is similar to [Laf18, Proposition 8.32] and the surjectivity is similar to *loc.cit.* Proposition 8.31 (*loc.cit.* is for a special case $\overline{x} = \Delta(\overline{\eta})$, where Δ is the diagonal morphism $\Delta : X \hookrightarrow X^I$). The proof uses the Eichler-Shimura relations and Lemma 2.3.2 below:

Lemma 2.3.2. *(consequence of* [Lau04] *Lemma 9.2.1) Let x be a point of $(\eta)^I$. The set $\{(\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{m_i})(x), (m_i)_{i\in I} \in \mathbb{N}^I\}$ is Zariski dense in X^I.*

Proof. The Zariski closure of this set is a closed subscheme Z of X^I, invariant by the partial Frobenius morphisms. If Z is not equal to X^I, by Lemma 1.3.1, Z is included in a finite union of vertical divisors. However, the image of x in X^I is not included in any vertical divisor. This is a contradiction. We deduce that $Z = X^I$. □

Proof of Proposition 2.3.1:

Injectivity: the proof is the same as Proposition 8.32 of [Laf18], except that we replace everywhere $\Delta(\overline{\eta})$ by $\overline{x}$ and replace everywhere $\Delta(\overline{v})$ by $\overline{y}$ (defined below). For the reader's convenience, we briefly recall the proof. Let $a \in \mathrm{Ker}(\mathfrak{sp}_{\overline{x}}^*)$. We want to prove that $a = 0$.

There exists μ_0 large enough and $\widetilde{a} \in \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{x}}$, such that a is the image of $\widetilde{a}$ in $\mathcal{H}_{I,W}\big|_{\overline{x}}$. We denote by x the image of $\overline{x}$ in $(X \smallsetminus N)^I$ and $\overline{\{x\}}$ the Zariski closure of x. Let Ω_0 be a dense open subscheme of $\overline{\{x\}}$ such that $\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\Omega_0}$ is lisse. Let y be a closed point in Ω_0. Let $\overline{y}$ be a geometric point over y and $\mathfrak{sp}_y : \overline{x} \to \overline{y}$ a specialization map over Ω_0. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{y}} & \xrightarrow[\simeq]{\mathfrak{sp}_y^*} & \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{x}} \\ \downarrow & & \downarrow \\ \mathcal{H}_{I,W}\Big|_{\overline{y}} & \xrightarrow{\mathfrak{sp}_y^*} & \mathcal{H}_{I,W}\Big|_{\overline{x}} \end{array} \tag{2.8}$$

The upper horizontal morphism is an isomorphism because $\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\Omega_0}$ is lisse. Thus there exists $\widetilde{b} \in \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{y}}$ such that $\widetilde{a} = \mathfrak{sp}_y^*(\widetilde{b})$. Let b be the image of $\widetilde{b}$ in $\mathcal{H}_{I,W}\big|_{\overline{y}}$. We have $a = \mathfrak{sp}_y^*(b)$.

Let y_i be the image of y by $(X \smallsetminus N)^I \xrightarrow{\mathrm{pr}_i} X \smallsetminus N$. Then $\times_{i\in I} y_i$ is a finite union of closed points containing y. Let $d_i = \deg(y_i)$. For any $(n_i)_{i\in I} \in \mathbb{N}^I$, we have $\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{d_i n_i}(\overline{y}) = \overline{y}$. (Note that in general $\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{d_i n_i}(\overline{x}) \neq \overline{x}$.) We have the partial Frobenius morphism

$$\prod_{i\in I} F_{\{i\}}^{d_i n_i} : \mathcal{H}_{I,W}\big|_{\overline{y}} = (\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{d_i n_i})^* \mathcal{H}_{I,W}\Big|_{\overline{y}} \to \mathcal{H}_{I,W}\big|_{\overline{y}}$$

Let

$$b_{(n_i)_{i\in I}} = \prod_{i\in I} F_{\{i\}}^{d_i n_i}(b) \in \mathcal{H}_{I,W}\big|_{\overline{y}} \quad \text{and} \quad a_{(n_i)_{i\in I}} = \mathfrak{sp}_y^*(b_{(n_i)_{i\in I}}) \in \mathcal{H}_{I,W}\big|_{\overline{x}}.$$

In particular, $b_{(0)_{i\in I}} = b$ and $a_{(0)_{i\in I}} = a$.

Let $d = \deg(y) = ppcm(\{d_i\}_{i\in I})$. Note that $\prod_{i\in I} \mathrm{Frob}_{\{i\}}$ is the total Frobenius morphism, thus the morphism

$$\prod_{i\in I} F_{\{i\}}^{dn} : \mathcal{H}_{I,W}\big|_{\overline{x}} \to \mathcal{H}_{I,W}\big|_{\overline{x}}$$

is bijective. We have

$$a_{(n_i + nd/d_i)_{i\in I}} = \prod_{i\in I} F_{\{i\}}^{dn}(a_{(n_i)_{i\in I}}). \tag{2.9}$$

[Laf18] Lemme 8.33 is still true if we replace everywhere $\Delta(\overline{\eta})$ by $\overline{x}$, replace everywhere $\Delta(\overline{v})$ by $\overline{y}$ and replace the the Eichler-Shimura relations [Laf18, Proposition 7.1] by Proposition 2.1.4. Thus we have:

(1) for all $j \in I$ and for all $(n_i)_{i\in I} \in \mathbb{N}^I$,

$$\sum_{\alpha=0}^{\mathrm{rk}\, M} (-1)^\alpha S_{\bigwedge^{\mathrm{rk}\, M-\alpha} M, y_j}(a_{(n_i+\alpha\delta_{i,j})_{i\in I}}) = 0 \ \text{ in } \mathcal{H}_{I,W}\big|_{\overline{x}}. \tag{2.10}$$

(2) Let $\mu_1 \geq \mu_0$ such that $\mathfrak{sp}_{\overline{x}}^*(\widetilde{a}) \in \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{\eta_I}}$ has zero image in $\mathcal{H}_{I,W}^{\leq\mu_1}\Big|_{\overline{\eta_I}}$. Let Ω_1 be a dense open subscheme of $(X \smallsetminus N)^I$ such that $\mathcal{H}_{I,W}^{\leq\mu_1}\Big|_{\Omega_1}$ is lisse. Then for every $(m_i)_{i\in I} \in \mathbb{N}^I$ such that $\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{d_i m_i}(x) \in \Omega_1$, we have $a_{(m_i)_{i\in I}} = 0$ in $\mathcal{H}_{I,W}\big|_{\overline{x}}$.

Note that the open subscheme

$$\bigcap_{(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\cdots,\mathrm{rk}\,M-1\}} \Big(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{d_i\alpha_i}\Big)^{-1}(\Omega_1) \tag{2.11}$$

is dense in X^I because Ω_1 is dense. By Lemma 1.3.1, there exists $(N_i)_{i\in I}\in\mathbb{N}^I$ such that $\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{d_iN_i}(x)$ is in (2.11). We deduce

$$\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{d_i(N_i+\alpha_i)}(x)\in\Omega_1 \text{ for all } (\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\cdots,\mathrm{rk}\,M-1\}.$$

By (2), we deduce that

$$a_{(N_i+\alpha_i)_{i\in I}}=0 \text{ for all } (\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\cdots,\mathrm{rk}\,M-1\}. \tag{2.12}$$

By (1), for every $j\in I$ and $(n_i)_{i\in I}\in\mathbb{N}^I$,

$$a_{(n_i+\mathrm{rk}\,M\delta_{i,j})_{i\in I}}=\sum_{\alpha=0}^{\mathrm{rk}\,M-1}(-1)^{\alpha+\mathrm{rk}\,M}S_{\bigwedge^{\mathrm{rk}\,M-\alpha}M,y_j}(a_{(n_i+\alpha\delta_{i,j})_{i\in I}}) \tag{2.13}$$

Using (2.12) and (2.13), by induction we deduce that

$$a_{(n_i)_{i\in I}}=0 \text{ for all } (n_i)_{i\in I}\in\mathbb{N}^I \text{ such that } n_i\geq N_i,\ \forall i\in I$$

Thus for $n\geq N_i$ for all $i\in I$, we have $a_{(nd/d_i)_{i\in I}}=0$. Then (2.9) implies $a_{(0)_{i\in I}}=0$. This proves the injectivity of $\mathfrak{sp}_{\overline{x}}^*$.

Surjectivity: To prove that $\mathfrak{sp}_{\overline{x}}^*$ is surjective, it is enough to prove that for every μ, we have $\mathfrak{M}_\mu\subset\mathrm{Im}(\mathfrak{sp}_{\overline{x}}^*)$.

As in the proof of Proposition 2.1.6, we define $\widetilde{\mathfrak{M}_\mu}$ to be the image of

$$\sum_{(n_i)_{i\in I}\in\mathbb{N}^I}(\otimes_{i\in I}\mathscr{H}_{G,v_i})\cdot\Big(\prod_{i\in I}F_{\{i\}}^{n_i}\big((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})^*\mathcal{H}_{I,W}^{\leq\mu}\big)\Big)\Big|_{\eta_I} \tag{2.14}$$

in $\mathcal{H}_{I,W}\big|_{\eta_I}$. It is a subsheaf of $\mathcal{H}_{I,W}\big|_{\eta_I}$. We have

$$\widetilde{\mathfrak{M}_\mu}\Big|_{\overline{\eta_I}}=\mathfrak{M}_\mu,$$

where $\mathfrak{M}_\mu$ is constructed in the proof of Proposition 2.1.6. Recall that the proof of Proposition 2.1.6 implies that there exists μ_0 large enough such that

$$\mathfrak{M}_\mu\subset\otimes_{i\in I}\mathscr{H}_{G,v_i}\cdot\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\overline{\eta_I}}. \tag{2.15}$$

Let Ω_0 be a dense open subscheme of $(X\smallsetminus N)^I$ such that $\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\Omega_0}$ is lisse. By Lemma 1.3.1, the set $\{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{m_i})(x),(m_i)_{i\in I}\in\mathbb{N}^I\}$ is Zariski dense in X^I. We deduce that there exists $(n_i)_{i\in I}\in\mathbb{N}^I$, such that $(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(x)\in\Omega_0$.

Let the specialization map

$$\Big(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i}\Big)\mathfrak{sp}_{\overline{x}}:\Big(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i}\Big)(\overline{\eta_I})\to\Big(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i}\Big)(\overline{x})$$

be the image of $\mathfrak{sp}_{\overline{x}}$ by $\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{x})} & \xrightarrow[\simeq]{((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*} & \mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})} \\ \downarrow & & \downarrow \\ \mathcal{H}_{I,W}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{x})} & \xrightarrow{((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*} & \mathcal{H}_{I,W}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})} \end{array} \tag{2.16}$$

The upper horizontal morphism is an isomorphism because $\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{\Omega_0}$ is lisse.

Since the action of the Hecke algebra is given by morphisms of sheaves, it commutes with $((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*$. We deduce that $\otimes_{i\in I}\mathscr{H}_{G,v_i}\cdot\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})}$ (view as image in $\mathcal{H}_{I,W}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})}$) is in the image of $((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*$.

Choose an isomorphism (not canonical) $\overline{\eta_I}\simeq(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})$, we deduce from (2.15) that

$$\widetilde{\mathfrak{M}_\mu}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})}\subset\otimes_{i\in I}\mathscr{H}_{G,v_i}\cdot\mathcal{H}_{I,W}^{\leq\mu_0}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})}. \tag{2.17}$$

So $\widetilde{\mathfrak{M}_\mu}\Big|_{(\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})}$ is in the image of $((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*$.

As in the proof of [Laf18] Proposition 8.31. we have a commutative diagram

(2.18)

$$\begin{array}{ccccc} \mathcal{H}_{I,W}\Big|_{(\prod\mathrm{Frob}_{\{i\}}^{n_i})(\overline{x})} & \xrightarrow{((\prod_{i\in I}\mathrm{Frob}_{\{i\}}^{n_i})\mathfrak{sp}_{\overline{x}})^*} & \mathcal{H}_{I,W}\Big|_{(\prod\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})} & \hookleftarrow & \widetilde{\mathfrak{M}_\mu}\Big|_{(\prod\mathrm{Frob}_{\{i\}}^{n_i})(\overline{\eta_I})} \\ \simeq\downarrow\prod_{i\in I}F_{\{i\}}^{n_i} & & \simeq\downarrow\prod_{i\in I}F_{\{i\}}^{n_i} & & \downarrow\simeq \\ \mathcal{H}_{I,W}\Big|_{\overline{x}} & \xrightarrow{\mathfrak{sp}_{\overline{x}}^*} & \mathcal{H}_{I,W}\Big|_{\overline{\eta_I}} & \hookleftarrow & \widetilde{\mathfrak{M}_\mu}\Big|_{\overline{\eta_I}} \end{array}$$

We deduce that $\widetilde{\mathfrak{M}_\mu}\Big|_{\overline{\eta_I}}$ is in the image of $\mathfrak{sp}_{\overline{x}}^*$.

□

Proposition 2.3.3. $\mathcal{H}_{I,W}\Big|_{(\overline{\eta})^I}$ *is a constant sheaf over* $(\overline{\eta})^I$.

Proof. (1) On the one hand, by Proposition 2.2.1, the action of $\mathrm{Ker}\,\Psi=\mathrm{Ker}(\mathrm{FWeil}(\eta_I,\overline{\eta_I})\to\mathrm{Weil}(\eta,\overline{\eta})^I)$ on $\mathcal{H}_{I,W}\Big|_{\overline{\eta_I}}$ is trivial.

(2) On the other hand, Proposition 2.3.1 says that $\mathcal{H}_{I,W}\Big|_{(\overline{\eta})^I}$ is ind-lisse over $(\overline{\eta})^I$.

A similar argument as in 1.3.3 implies that the action of $\pi_1((\overline{\eta})^I,\overline{\eta_I})$ on $\mathcal{H}_{I,W}\Big|_{\overline{\eta_I}}$ is trivial. We deduce the result. □

2.4. **Constancy over** $(\overline{\eta})^{I_1}\times_{\overline{\mathbb{F}_q}}\overline{u}$**.** To prove the smoothness result in the next section, we need to prove Proposition 2.4.5. When I_2 is empty, we recover Proposition 2.3.3.

2.4.1. Let I be a disjoint union $I_1 \sqcup I_2$. Let u be a closed point of $(X \smallsetminus N)^{I_2}$ and $\overline{u}$ a geometric point over u.

Let $\mathcal{F}$ be an ind-constructible Λ-sheaf over $(\eta)^{I_1} \times_{\mathbb{F}_q} u$, equipped with an action of the partial Frobenius morphisms, i.e. for every $i \in I_1$, an isomorphism $F_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ and an isomorphism $F_{I_2} : \mathrm{Frob}_{I_2}^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$, commuting with each other and whose composition is the total Frobenius isomorphism $\mathrm{Frob}^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ over $(\eta)^{I_1} \times_{\mathbb{F}_q} u$. Then the fiber $\mathcal{F}|_{\overline{\eta_{I_1}} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ is equipped with an action of $\mathrm{FWeil}(\eta_{I_1}, \overline{\eta_{I_1}})$.

In particuler, the fiber of the cohomology sheaf $\mathcal{H}_{I,W}|_{\overline{\eta_{I_1}} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ is equipped with a continuous action of $\mathrm{FWeil}(\eta_{I_1}, \overline{\eta_{I_1}})$.

Proposition 2.4.2. *$\mathcal{H}_{I,W}|_{\overline{\eta_{I_1}} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ is an increasing union of sub Λ-modules $\mathfrak{M}$ which are stable under the action of $\mathrm{FWeil}(\eta_{I_1}, \overline{\eta_{I_1}})$, and for which there exists a family $(v_i)_{i\in I}$ of closed points in $X \smallsetminus N$ (depending on $\mathfrak{M}$) such that $\mathfrak{M}$ is stable under the action of $\otimes_{i\in I} \mathscr{H}_{G,v_i}$ and is of finite type as module over $\otimes_{i\in I} \mathscr{H}_{G,v_i}$.*

Proof. Similar to Proposition 2.1.6. □

Proposition 2.4.3. *The action of $\mathrm{FWeil}(\eta_{I_1}, \overline{\eta_{I_1}})$ on $\mathcal{H}_{I,W}|_{\overline{\eta_{I_1}} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ factors through $\mathrm{Weil}(\eta, \overline{\eta})^{I_1}$.*

Proof. Similar to Proposition 2.2.1. □

Proposition 2.4.4. *$\mathcal{H}_{I,W}|_{(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ is ind-lisse over $(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} \overline{u}$.*

Proof. Similar to Proposition 2.3.1. □

Proposition 2.4.5. *$\mathcal{H}_{I,W}|_{(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} \overline{u}}$ is constant over $(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} \overline{u}$.*

Proof. Similar to Proposition 2.3.3. □

2.4.6. Let s be a closed point of $X \smallsetminus N$ and $\overline{s}$ a geometric point over s. Let

$$(\overline{s})^{I_2} := \overline{s} \times_{\overline{\mathbb{F}_q}} \cdots \times_{\overline{\mathbb{F}_q}} \overline{s}.$$

Then $(\overline{s})^{I_2}$ is a special case of $\overline{u}$, for $u = \Delta(s)$, where $\Delta : X \hookrightarrow X^{I_2}$ is the diagonal inclusion. By Proposition 2.4.5, $\mathcal{H}_{I,W}|_{(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} (\overline{s})^{I_2}}$ is constant over $(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} (\overline{s})^{I_2}$.

3. Smoothness of the cohomology sheaves over $(X \smallsetminus N)^I$

The goal of this section is to prove Theorem 3.2.3 and Theorem 3.3.1. We refer to [Xue22] for the case when I is a singleton for illustration (in this case we only need to consider S a henselian trait, i.e. spectrum of a henselian DVR). The general case that we prove in this section is similar.

3.1. **Some preparations.**

3.1.1. Let S be a local henselian ring over a perfect field k (not necessarily of dimension one). Let $s = \mathrm{Spec}\, k$ be the closed point and $\delta = \mathrm{Spec}\, K$ the generic point. Fix an algebraic closure $\overline{K}$ of K. We denote by $\overline{\delta} = \mathrm{Spec}\, \overline{K}$. It will be enough for us to consider only the case where k is separably closed, i.e. we assume $\overline{k} = k$.

3.1.2. Let I be a finite set. Let $(\overline{x_i})_{i\in I}$ be a family of geometric points of S such that $\overline{x_i} \in \{\overline{s}, \overline{\delta}\}$. We denote by $\times_{i\in I}\overline{x_i}$ the fiber product over $\mathrm{Spec}\, k$. As in 1.1.5, $\times_{i\in I}\overline{x_i}$ is an integral scheme over $\mathrm{Spec}\, k$.

Definition 3.1.3. Let $\mathcal{G}$ be an ind-constructible Λ-sheaf over S^I (product of I-copies of S over k). We say that $\mathcal{G}$ is a pseudo-product [2] if for any family $(\overline{x_i})_{i\in I}$ of geometric points of S such that $\overline{x_i} \in \{\overline{s}, \overline{\delta}\}$, the restriction $\mathcal{G}\big|_{\times_{i\in I}\overline{x_i}}$ is a constant sheaf over $\times_{i\in I}\overline{x_i}$.

Notation 3.1.4. *For pseudo-product sheaf $\mathcal{G}$, we denote*

$$\mathcal{G}\big|_{\times_{i\in I}\overline{x_i}} := \Gamma(\times_{i\in I}\overline{x_i}, \mathcal{G}).$$

Example 3.1.5. If $\mathcal{G} = \boxtimes_{i\in I}\mathcal{F}_i$ where every $\mathcal{F}_i$ is an ind-constructible Λ-sheaf over S, then $\mathcal{G}$ is a pseudo-product.

Example 3.1.6. For any geometric point $\overline{v}$ of $X \smallsetminus N$, let $S = (X \smallsetminus N)_{(\overline{v})}$ be the strict henselization of $X \smallsetminus N$ at $\overline{v}$. Let $\overline{s} = s = \overline{v}$ and $\overline{\delta} = \overline{\eta}$. Then by 2.4.6, $\mathcal{H}_{I,W}\big|_{S^I}$ is a pseudo-product.

3.1.7. Our choice of $\overline{\delta} \to S$ is a specialization map $\mathfrak{sp} : \overline{\delta} \to \overline{s}$.

Let $\mathcal{G}$ be an ind-constructible Λ-sheaf over S, then $\mathfrak{sp} : \overline{\delta} \to \overline{s}$ induces a morphism (by restriction because $\mathcal{G}\big|_{\overline{s}} = \Gamma(S, \mathcal{G})$)

$$\phi_{0,1} : \mathcal{G}\big|_{\overline{s}} \to \mathcal{G}\big|_{\overline{\delta}}.$$

3.1.8. Let S^+ be the normalization of $\overline{\delta}$ in S. Then S^+ is still a local henselian ring, with closed point $s = \overline{s}$ and generic point $\overline{\delta}$. Let $\mathcal{G}$ be a pseudo-product sheaf over S^I. It is still a pseudo-product sheaf over $(S^+)^I$.

3.1.9. Let $\overline{u}$ be a geometric point over some $\times_{i\in I}\overline{x_i}$ in $(S^+)^I$ and $\overline{v}$ a geometric point over some other $\times_{i\in I}\overline{x_i}'$ in $(S^+)^I$, with $\overline{x_i}, \overline{x_i}' \in \{\overline{s}, \overline{\delta}\}$. If $\overline{u}$ is a specialization of $\overline{v}$ (i.e. there exists a specialization map $\overline{v} \to \overline{u}$, i.e. a morphism $\overline{v} \to (S^+)^I_{(\overline{u})}$), then we will construct a canonical morphism (for the moment it depends on $\overline{v}$ and $\overline{u}$, but does not depend on the choice of specialization map)

$$\phi_{\overline{u},\overline{v}} : \mathcal{G}\big|_{\times_{i\in I}\overline{x_i}} \to \mathcal{G}\big|_{\times_{i\in I}\overline{x_i}'}. \tag{3.1}$$

Here is the construction: denote by v the image of $\overline{v}$ in $(S^+)^I$. Note that $\times_{i\in I}\overline{x_i}$ is a subscheme of $(S^+)^I$, thus v is a point over $\times_{i\in I}\overline{x_i}$.

[2] This is a condition, not a structure. I do not know if this condition is equivalent to being in the essential image of $(\mathrm{Shv}(S))^{\boxtimes I}$.

For any specialization map $\mathfrak{a} : \overline{v} \to \overline{u}$, it induces a morphism $\mathfrak{a}^* : \mathcal{G}\big|_{\overline{u}} \to \mathcal{G}\big|_{\overline{v}}$. The group $\mathrm{Gal}(\overline{v}/v)$ acts transitively on the set of specialization maps $\{\overline{v} \to \overline{u}\}$ and acts on the set of induced morphisms $\mathcal{G}\big|_{\overline{u}} \to \mathcal{G}\big|_{\overline{v}}$ by acting on $\mathcal{G}\big|_{\overline{v}}$.

Since $\mathcal{G}$ is a pseudo-product, it is constant over $\times_{i\in I}\overline{x_i}$, so is constant over v. The action of $\mathrm{Gal}(\overline{v}/v)$ on $\mathcal{G}\big|_{\overline{v}}$ is trivial. Thus the morphism $\mathcal{G}\big|_{\overline{u}} \to \mathcal{G}\big|_{\overline{v}}$ does not depend on the choice of specialization map. Moreover, since $\mathcal{G}$ is a pseudo-product, we can identify $\mathcal{G}\big|_{\overline{u}} = \mathcal{G}\big|_{\times_{i\in I}\overline{x_i}}$ and $\mathcal{G}\big|_{\overline{v}} = \mathcal{G}\big|_{\times_{i\in I}\overline{x_i}'}$. We obtain morphism (3.1).

When I is a singleton, we recover 3.1.7.

Example 3.1.10. Let $I = \{1, 2\}$, $\overline{x_1} \times \overline{x_2} = \overline{\delta} \times \overline{s}$, $\overline{x_1}' \times \overline{x_2}' = \overline{\delta} \times \overline{\delta}$. Let $\overline{u} = \overline{\delta} \times \overline{s}$ and $\overline{v} = \overline{\delta_2}$, where δ_2 is the generic point of $S \times S$. Then (3.1) is a morphism $\mathcal{G}\big|_{\overline{\delta}\times\overline{s}} \to \mathcal{G}\big|_{\overline{\delta}\times\overline{\delta}}$.

3.1.11. The canonical morphism (3.1) is compatible with the composition. Let $\overline{w}$ be a geometric point over some $\times_{i\in I}\overline{x_i}''$ in $(S^+)^I$, with $\overline{x_i}'' \in \{\overline{s}, \overline{\delta}\}$. Suppose that $\overline{u}$ is a specialization of $\overline{v}$ and $\overline{v}$ is a specialization of $\overline{w}$. Then we have

$$\phi_{\overline{v},\overline{w}} \circ \phi_{\overline{u},\overline{v}} = \phi_{\overline{u},\overline{w}}$$

To see this, we choose specialization maps such that the following diagram of specialization maps commutes (it is enough to choose $\overline{w} \to \overline{u}$ to be the composition of $\overline{w} \to \overline{v}$ and $\overline{v} \to \overline{u}$):

$$\begin{array}{ccc} \overline{w} & \longrightarrow & \overline{v} \\ & \searrow & \downarrow \\ & & \overline{u} \end{array}$$

It induces a commutative diagram

$$\begin{array}{ccc} \mathcal{G}\big|_{\overline{w}} & \longleftarrow & \mathcal{G}\big|_{\overline{v}} \\ & \nwarrow & \uparrow \\ & & \mathcal{G}\big|_{\overline{u}} \end{array} \tag{3.2}$$

By 3.1.9, these morphisms do not depend on the choice of specialization maps. We identify (3.2) with the following commutative diagram of canonical morphisms:

$$\begin{array}{ccc} \mathcal{G}\big|_{\times\overline{x_i}''} & \xleftarrow{\phi_{\overline{v},\overline{w}}} & \mathcal{G}\big|_{\times\overline{x_i}'} \\ & \nwarrow^{\phi_{\overline{u},\overline{w}}} & \uparrow{\scriptstyle \phi_{\overline{u},\overline{v}}} \\ & & \mathcal{G}\big|_{\times\overline{x_i}} \end{array}$$

3.1.12. Now we prove that (3.1) constructed in 3.1.9 does not depend on $\overline{u}$ and $\overline{v}$. Let $\overline{u}'$ be a geometric generic point of $\times_{i\in I}\overline{x_i}$ and $\overline{v}'$ be a geometric generic point of $\times_{i\in I}\overline{x_i}'$. Then there exists specialization maps $\overline{u}' \to \overline{u}$ and $\overline{v}' \to \overline{v}$. Since there exists a specialization map $\overline{v} \to \overline{u}$, there exists a specialization map $\overline{v}' \to \overline{u}'$.

By 3.1.11, we have

$$\psi_{\overline{v},\overline{v}'} \circ \psi_{\overline{u},\overline{v}} = \psi_{\overline{u},\overline{v}'} = \psi_{\overline{u}',\overline{v}'} \circ \psi_{\overline{u},\overline{u}'}$$

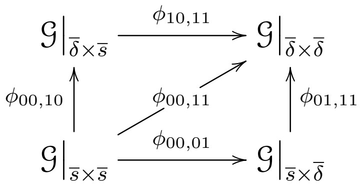
Since $\mathcal{G}$ is constant over $\times_{i \in I} \overline{x_i}$ (resp. $\times_{i \in I} \overline{x_i}'$), we have $\phi_{\overline{u},\overline{u}'} = \mathrm{Id}$ (resp. $\phi_{\overline{v},\overline{v}'} = \mathrm{Id}$. We deduce $\phi_{\overline{u},\overline{v}} = \phi_{\overline{u}',\overline{v}'}$.

So $\phi_{\overline{u},\overline{v}}$ constructed in 3.1.9 does not depend on the choice of geometric points $\overline{u}$ and $\overline{v}$. We obtain a canonical morphism

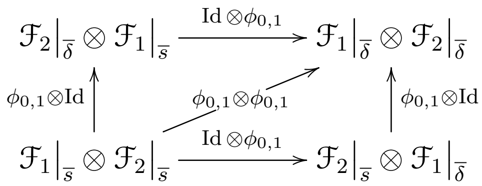
$$\phi_{\times \overline{x_i}, \times \overline{x_i}'} : \mathcal{G}\big|_{\times_{i \in I} \overline{x_i}} \to \mathcal{G}\big|_{\times_{i \in I} \overline{x_i}'}$$

Construction 3.1.13. *Let $\mathcal{G}$ be an ind-constructible Λ-sheaf over $S \times S$ which is a pseudo-product. Applying 3.1.9-3.1.12 to $S^+ \times S^+$, we construct the following canonical morphisms which form a commutative diagram:*

(3.3)

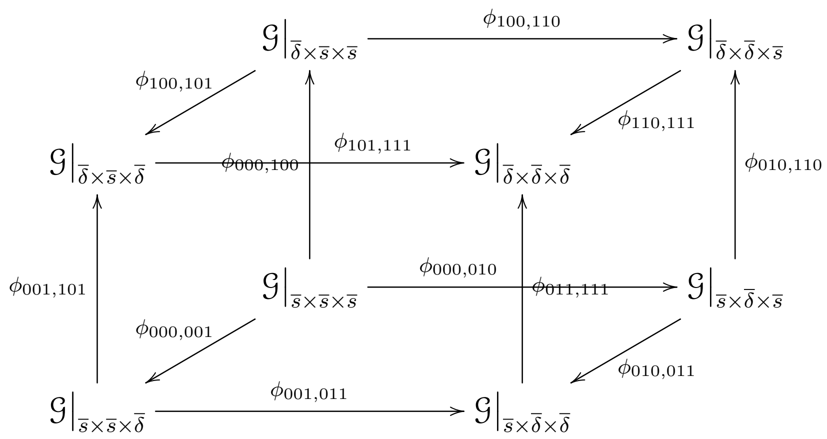


Example 3.1.14. In Construction 3.1.13, when $\mathcal{G} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$, diagram (3.3) coincides with



Construction 3.1.15. *Let $\mathcal{G}$ be an ind-constructible Λ-sheaf over $S \times S \times S$ which is a pseudo-product. Applying 3.1.9-3.1.12 to $S^+ \times S^+ \times S^+$, we construct the following canonical morphisms which form a commutative diagram:*

(3.4)



3.1.16. The morphism induced by a specialization map is functorial for morphism of sheaves $\mathcal{G}_1 \to \mathcal{G}_2$. We deduce that the canonical morphism constructed in 3.1.9 is functorial for morphism of sheaves. In particular, Constructions 3.1.13 and 3.1.15 are functorial for morphism of sheaves.

3.1.17. Let $\mathcal{G}$ be a pseudo-product sheaf over $S \times S \times S$. Consider the partial diagonal morphism

$$S \times S \xrightarrow{(\Delta^{\{1,2\}}, \mathrm{Id})} S \times S \times S. \tag{3.5}$$

The restriction $\mathcal{G}\big|_{\Delta^{\{1,2\}}(S) \times S}$ is a pseudo-product sheaf over $S \times S$. Then (3.3) for the sheaf $\mathcal{G}\big|_{\Delta^{\{1,2\}}(S) \times S}$ coincides with the following commutative sub-diagram of (3.4):

$$\begin{array}{ccc} \mathcal{G}\big|_{\overline{\delta} \times \overline{\delta} \times \overline{s}} & \xrightarrow{\phi_{10,11} = \phi_{110,111}} & \mathcal{G}\big|_{\overline{\delta} \times \overline{\delta} \times \overline{\delta}} \\ {\scriptstyle \phi_{00,10} = \phi_{000,110}} \uparrow & & \uparrow {\scriptstyle \phi_{01,11} = \phi_{001,111}} \\ \mathcal{G}\big|_{\overline{s} \times \overline{s} \times \overline{s}} & \xrightarrow{\phi_{00,01} = \phi_{000,001}} & \mathcal{G}\big|_{\overline{s} \times \overline{s} \times \overline{\delta}} \end{array} \tag{3.6}$$

In fact, this is because by 3.1.12, to construct the morphisms in (3.6), we can choose geometric points over the subschemes $\Delta^{\{1,2\}}(\overline{s}) \times \overline{s}$ (resp. $\Delta^{\{1,2\}}(\overline{\delta}) \times \overline{s}$, $\Delta^{\{1,2\}}(\overline{s}) \times \overline{\delta}$, $\Delta^{\{1,2\}}(\overline{\delta}) \times \overline{\delta}$) of $\overline{s} \times \overline{s} \times \overline{s}$ (resp. $\overline{\delta} \times \overline{\delta} \times \overline{s}$, $\overline{s} \times \overline{s} \times \overline{\delta}$, $\overline{\delta} \times \overline{\delta} \times \overline{\delta}$).

Similarly, consider the partial diagonal morphism

$$S \times S \xrightarrow{(\mathrm{Id}, \Delta^{\{2,3\}})} S \times S \times S. \tag{3.7}$$

The restriction $\mathcal{G}\big|_{S \times \Delta^{\{2,3\}}(S)}$ is a pseudo-product sheaf over $S \times S$. Then (3.3) for the sheaf $\mathcal{G}\big|_{S \times \Delta^{\{2,3\}}(S)}$ coincides with the following commutative sub-diagram of (3.4):

$$\begin{array}{ccc} \mathcal{G}\big|_{\overline{\delta} \times \overline{s} \times \overline{s}} & \xrightarrow{\phi_{10,11} = \phi_{100,111}} & \mathcal{G}\big|_{\overline{\delta} \times \overline{\delta} \times \overline{\delta}} \\ {\scriptstyle \phi_{00,10} = \phi_{000,100}} \uparrow & & \uparrow {\scriptstyle \phi_{01,11} = \phi_{011,111}} \\ \mathcal{G}\big|_{\overline{s} \times \overline{s} \times \overline{s}} & \xrightarrow{\phi_{00,01} = \phi_{000,011}} & \mathcal{G}\big|_{\overline{s} \times \overline{\delta} \times \overline{\delta}} \end{array}$$

3.2. **Smoothness of $\mathcal{H}_{I,W}$.** Let I be a finite set and $W \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$.

3.2.1. Let $I_0 = I_1 = I_2 = I_3 = I$. We denote by

$$\Delta^{I_1 \sqcup I_2 \sqcup I_3} : (X \smallsetminus N)^I \to (X \smallsetminus N)^{I_1} \times (X \smallsetminus N)^{I_2} \times (X \smallsetminus N)^{I_3},$$
$$(x_i)_{i \in I} \mapsto \big((x_i)_{i \in I_1}, (x_i)_{i \in I_2}, (x_i)_{i \in I_3}\big)$$

$$\Delta^{I_1 \sqcup I_2} : (X \smallsetminus N)^I \to (X \smallsetminus N)^{I_1} \times (X \smallsetminus N)^{I_2}, \quad (x_i)_{i \in I} \mapsto \big((x_i)_{i \in I_1}, (x_i)_{i \in I_2}\big)$$

$$\Delta^{I_2 \sqcup I_3} : (X \smallsetminus N)^I \to (X \smallsetminus N)^{I_2} \times (X \smallsetminus N)^{I_3}, \quad (x_i)_{i \in I} \mapsto \big((x_i)_{i \in I_2}, (x_i)_{i \in I_3}\big)$$

We denote by $\mathbf{1}$ the trivial representation of $\widehat{G}^I$. Let unit : $\mathbf{1} \to W^* \otimes W$ be the canonical morphism. As in [Laf18] Section 5, we define the creation operator

$\mathcal{C}^\sharp := \mathcal{C}^{\sharp, I_2 \sqcup I_3}_{\mathrm{unit}}$ (creating legs $I_2 \sqcup I_3$) to be the composition of morphisms of sheaves over $(X \smallsetminus N)^I \times (X \smallsetminus N)^I$:

$$\begin{aligned} \mathcal{H}_{I_1, W} \boxtimes \Lambda_{(X \smallsetminus N)^I} \xrightarrow{\sim} \mathcal{H}_{I_1 \sqcup I_0, W \boxtimes \mathbf{1}} \xrightarrow{\mathcal{H}(\mathrm{Id}_W \boxtimes \mathrm{unit})} \mathcal{H}_{I_1 \sqcup I_0, W \boxtimes (W^* \otimes W)} \\ \xrightarrow[\sim]{\chi^{-1}_{I_2 \sqcup I_3}} \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\Big|_{(X \smallsetminus N)^{I_1} \times \Delta^{I_2 \sqcup I_3}\left((X \smallsetminus N)^I\right)} \end{aligned} \tag{3.8}$$

where $\mathcal{H}(\mathrm{Id}_W \boxtimes \mathrm{unit})$ is by the functoriality associated to $\mathrm{Id}_W \boxtimes \mathrm{unit} : W \boxtimes \mathbf{1} \to W \boxtimes (W^* \otimes W)$, $\chi_{I_2 \sqcup I_3}$ is the fusion isomorphism ([Laf18] Proposition 4.12) associated to the map

$$I_1 \sqcup I_2 \sqcup I_3 \twoheadrightarrow I_1 \sqcup I_0$$

sending I_1 to I_1 by identity, I_2 to I_0 by identity and I_3 to I_0 by identity.

Let $\mathrm{ev} : W \otimes W^* \to \mathbf{1}$ be the evaluation map. We define the annihilation operator $\mathcal{C}^\flat := \mathcal{C}^{\flat, I_1 \sqcup I_2}_{\mathrm{ev}}$ (annihilating legs $I_1 \sqcup I_2$) to be the composition of morphisms of sheaves over $(X \smallsetminus N)^I \times (X \smallsetminus N)^I$:

$$\begin{aligned} \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{I_1 \sqcup I_2}\left((X \smallsetminus N)^I\right) \times (X \smallsetminus N)^{I_3}} \xrightarrow[\sim]{\chi_{I_1 \sqcup I_2}} \mathcal{H}_{I_0 \sqcup I_3, (W \otimes W^*) \boxtimes W} \\ \xrightarrow{\mathcal{H}(\mathrm{ev} \boxtimes \mathrm{Id}_W)} \mathcal{H}_{I_0 \sqcup I_3, \mathbf{1} \boxtimes W} \xrightarrow{\sim} \Lambda_{(X \smallsetminus N)^I} \boxtimes \mathcal{H}_{I_3, W} \end{aligned} \tag{3.9}$$

where $\mathcal{H}(\mathrm{ev} \boxtimes \mathrm{Id}_W)$ is by the functoriality associated to $\mathrm{ev} \boxtimes \mathrm{Id}_W : (W \otimes W^*) \boxtimes W \to \mathbf{1} \boxtimes W$, $\chi_{I_1 \sqcup I_2}$ is the fusion isomorphism associated to the map

$$I_1 \sqcup I_2 \sqcup I_3 \twoheadrightarrow I_0 \sqcup I_3$$

sending I_1 to I_0 by identity, I_2 to I_0 by identity and I_3 to I_3 by identity.

Lemma 3.2.2. *("Zorro" lemma,* [Laf18] *(6.18)) The composition of morphisms of sheaves over $(X \smallsetminus N)^I$:*

$$\mathcal{H}_{I_1, W} \xrightarrow{\mathcal{C}^\sharp} \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{I_1 \sqcup I_2 \sqcup I_3}\left((X \smallsetminus N)^I\right)} \xrightarrow{\mathcal{C}^\flat} \mathcal{H}_{I_3, W} \tag{3.10}$$

is the identity.

Proof. By the fusion property of Satake sheaves, we have

$$\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{I_1 \sqcup I_2 \sqcup I_3}((X \smallsetminus N)^I)} \simeq \mathcal{H}_{I, W \otimes W^* \otimes W}.$$

The composition of morphisms of vector spaces

$$W \xrightarrow{(\mathrm{Id}, \mathrm{unit})} W \otimes W^* \otimes W \xrightarrow{(\mathrm{ev}, \mathrm{Id})} W$$

is identity. The lemma follows from the functoriality on W. □

Theorem 3.2.3. *The ind-constructible Λ-sheaf $\mathcal{H}_{I,W}$ is ind-lisse over $(X \smallsetminus N)^I$.*

Proof. By Lemma A.0.3, it is enough to prove that for any geometric point $\overline{x}$ of $(X \smallsetminus N)^I$ and any specialization map $\mathfrak{sp}_{\overline{x}} : \overline{\eta_I} \to \overline{x}$, the induced morphism

$$\mathfrak{sp}^*_{\overline{x}} : \mathcal{H}_{I,W}\big|_{\overline{x}} \to \mathcal{H}_{I,W}\big|_{\overline{\eta_I}} \tag{3.11}$$

is an isomorphism.

Let γ be the composition of the following morphisms:

$$
\begin{aligned}
\mathcal{H}_{I_1,W}\big|_{\overline{\eta_I}} \otimes \Lambda\big|_{\overline{x}} &\xrightarrow{\mathcal{C}^{\sharp}} \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\Delta^{I_2\sqcup I_3}(\overline{x})} \\
&\overset{(\alpha)}{=} \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\overline{x}\times\overline{x}} \\
&\xrightarrow{\phi_{100,110}} \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\overline{\eta_I}\times\overline{x}} \\
&\overset{(\beta)}{=} \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\Delta^{I_1\sqcup I_2}(\overline{\eta_I})\times\overline{x}} \\
&\xrightarrow{\mathcal{C}^{\flat}} \Lambda\big|_{\overline{\eta_I}} \otimes \mathcal{H}_{I_3,W}\big|_{\overline{x}}.
\end{aligned} \tag{3.12}
$$

Here is the construction of the morphism $\phi_{100,110}$: let $S=((X\smallsetminus N)^I)_{(\overline{x})}$ be the strict henselization of $(X\smallsetminus N)^I$ at $\overline{x}$, its closed point is $\overline{x}$. The specialization map $\mathfrak{sp}_{\overline{x}}:\overline{\eta_I}\to\overline{x}$ is a morphism $\overline{\eta_I}\to S$. By Proposition 2.4.5, $\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}$ is constant over $(\overline{\eta})^{I_1\sqcup I_2\sqcup I_3}$ (resp. $(\overline{\eta})^{I_1\sqcup I_2}\times\overline{x}$, $(\overline{\eta})^{I_1\sqcup I_3}\times\overline{x}$, $(\overline{\eta})^{I_2\sqcup I_3}\times\overline{x}$), thus constant over $\overline{\eta_I}\times\overline{\eta_I}\times\overline{\eta_I}$ (resp. $\overline{\eta_I}\times\overline{\eta_I}\times\overline{x}$, $\overline{\eta_I}\times\overline{x}\times\overline{\eta_I}$, $\overline{x}\times\overline{\eta_I}\times\overline{\eta_I}$). So $\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}$ is a pseudo-product over $S\times S\times S$. We apply Construction 3.1.15 to $\mathcal{G}=\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}$, $\overline{s}=s=\overline{x}$, $\overline{\delta}=\overline{\eta_I}$, δ the image of $\overline{\eta_I}$ in S.

The equality (α) is evident and the equality (β) is because $\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}$ is constant over $\overline{\eta_I}\times\overline{\eta_I}\times\overline{x}$.

By Lemma 3.2.4 and Lemma 3.2.5 below, γ is the inverse of $\mathfrak{sp}^*$. □

Lemma 3.2.4. *The composition $\gamma\circ\mathfrak{sp}^*$ is the identity.*

Proof. The following diagram is commutative:
(3.13)

$$
\begin{array}{ccc}
\mathcal{H}_{I_1,W}\big|_{\overline{x}} \otimes \Lambda\big|_{\overline{x}} & \xrightarrow{\quad\mathfrak{sp}^*\quad} & \mathcal{H}_{I_1,W}\big|_{\overline{\eta_I}} \otimes \Lambda\big|_{\overline{x}} \\
\downarrow{\scriptstyle \mathcal{C}^{\sharp}} & & \downarrow{\scriptstyle \mathcal{C}^{\sharp}} \\
\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{x}\times\Delta^{I_2\sqcup I_3}(\overline{x})} & \xrightarrow{\phi_{00,10}} & \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\Delta^{I_2\sqcup I_3}(\overline{x})} \\
\| & & \| \\
\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{x}\times\overline{x}\times\overline{x}} & \xrightarrow{\phi_{000,100}} & \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\overline{x}\times\overline{x}} \\
 & \searrow{\scriptstyle \phi_{000,110}} & \downarrow{\scriptstyle \phi_{100,110}} \\
\| & & \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\overline{\eta_I}\times\overline{\eta_I}\times\overline{x}} \\
 & & \| \\
\mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\Delta^{I_1\sqcup I_2}(\overline{x})\times\overline{x}} & \xrightarrow{\phi_{00,10}} & \mathcal{H}_{I_1\sqcup I_2\sqcup I_3, W\boxtimes W^*\boxtimes W}\big|_{\Delta^{I_1\sqcup I_2}(\overline{\eta_I})\times\overline{x}} \\
\downarrow{\scriptstyle \mathcal{C}^{\flat}} & & \downarrow{\scriptstyle \mathcal{C}^{\flat}} \\
\Lambda\big|_{\overline{x}} \otimes \mathcal{H}_{I_3,W}\big|_{\overline{x}} & \xrightarrow{\quad\mathrm{Id}\quad} & \Lambda\big|_{\overline{\eta_I}} \otimes \mathcal{H}_{I_3,W}\big|_{\overline{x}}
\end{array}
$$

The first square is commutative because Construction 3.1.13 commutes with the morphism of sheaves $\mathcal{C}^{\sharp}$. The second square is commutative because we identify

$\phi_{00,10}$ with $\phi_{000,100}$ as in 3.1.17 for $\mathcal{G} = \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}$ and $\Delta^{I_2 \sqcup I_3}$. The middle triangle is commutative by Construction 3.1.15. The next square is commutative because we identify $\phi_{000,110}$ with $\phi_{00,10}$ as in 3.1.17 for $\mathcal{G} = \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}$ and $\Delta^{I_1 \sqcup I_2}$. The last square is commutative because Construction 3.1.13 commutes with the morphism of sheaves $\mathcal{C}^\flat$.

The composition of right vertical morphisms is γ. By Lemma 3.2.2, the composition of the left vertical morphisms is the identity. Since Λ is a constant sheaf, the lower horizontal line is identity. So $\gamma \circ \mathfrak{sp}^*$ is the identity. □

Lemma 3.2.5. *The composition* $\mathfrak{sp}^* \circ \gamma$ *is the identity.*

Proof. The following diagram is commutative
(3.14)

$$\begin{array}{ccc}
\mathcal{H}_{I_1,W}\big|_{\overline{\eta_I}} \otimes \Lambda\big|_{\overline{x}} & \xrightarrow{\mathrm{Id}} & \mathcal{H}_{I_1,W}\big|_{\overline{\eta_I}} \otimes \Lambda\big|_{\overline{\eta_I}} \\
\downarrow \mathcal{C}^\sharp & & \downarrow \mathcal{C}^\sharp \\
\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \Delta^{I_2 \sqcup I_3}(\overline{x})} & \xrightarrow{\phi_{10,11}} & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \Delta^{I_2 \sqcup I_3}(\overline{\eta_I})} \\
\downarrow = & & \downarrow = \\
\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \overline{x} \times \overline{x}} & \xrightarrow{\phi_{100,111}} & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \overline{\eta_I} \times \overline{\eta_I}} \\
\downarrow \phi_{100,110} & \nearrow \phi_{110,111} & \\
\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \overline{\eta_I} \times \overline{x}} & & \downarrow = \\
\downarrow = & & \\
\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\Delta^{I_1 \sqcup I_2}(\overline{\eta_I}) \times \overline{x}} & \xrightarrow{\phi_{10,11}} & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\Delta^{I_1 \sqcup I_2}(\overline{\eta_I}) \times \overline{\eta_I}} \\
\downarrow \mathcal{C}^\flat & & \downarrow \mathcal{C}^\flat \\
\Lambda\big|_{\overline{\eta_I}} \otimes \mathcal{H}_{I_3,W}\big|_{\overline{x}} & \xrightarrow{\mathfrak{sp}^*} & \Lambda\big|_{\overline{\eta_I}} \otimes \mathcal{H}_{I_3,W}\big|_{\overline{\eta_I}}
\end{array}$$

The first square is commutative because Construction 3.1.13 commutes with the morphism of sheaves $\mathcal{C}^\sharp$. The second square is commutative because we identify $\phi_{10,11}$ with $\phi_{100,111}$ as in 3.1.17 for $\mathcal{G} = \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}$ and $\Delta^{I_2 \sqcup I_3}$. The middle triangle is commutative by Construction 3.1.15. The next square is commutative because we identify $\phi_{110,111}$ with $\phi_{10,11}$ as in 3.1.17 for $\mathcal{G} = \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}$ and $\Delta^{I_1 \sqcup I_2}$. The last square is commutative because Construction 3.1.13 commutes with the morphism of sheaves $\mathcal{C}^\flat$.

Since $\mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \overline{\eta_I} \times \overline{\eta_I}}$ is constant, we have canonical isomorphisms:

$$\begin{array}{ccc} & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \Delta^{I_2 \sqcup I_3}(\overline{\eta_I})} & \\ \swarrow^{=} & & \searrow^{=} \\ \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\overline{\eta_I} \times \overline{\eta_I} \times \overline{\eta_I}} & & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\Delta^{I_1 \sqcup I_2 \sqcup I_3}(\overline{\eta_I})} \\ \searrow_{=} & & \swarrow_{=} \\ & \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\Delta^{I_1 \sqcup I_2}(\overline{\eta_I}) \times \overline{\eta_I}} & \end{array}$$

Using this, we identify the composition of right vertical morphisms with

$$\mathcal{H}_{I_1, W}\big|_{\overline{\eta_I}} \otimes \Lambda\big|_{\overline{\eta_I}} \xrightarrow{\mathcal{C}^\sharp} \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W \boxtimes W^* \boxtimes W}\big|_{\Delta^{I_1 \sqcup I_2 \sqcup I_3}(\overline{\eta_I})} \xrightarrow{\mathcal{C}^\flat} \Lambda\big|_{\overline{\eta_I}} \otimes \mathcal{H}_{I_3, W}\big|_{\overline{\eta_I}}.$$

By Lemma 3.2.2, the composition is the identity.

The composition of left vertical morphisms is γ. Since Λ is a constant sheaf, the upper horizontal line is identity. So $\mathfrak{sp}^* \circ \gamma$ is the identity. □

Corollary 3.2.6. *(of Theorem 3.2.3) The action of* $\mathrm{Weil}(\eta, \overline{\eta})^I$ *on* $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ *(defined in Proposition 2.2.1) factors through* $\mathrm{Weil}(X \smallsetminus N, \overline{\eta})^I$*.*

Proof. Let $K := \mathrm{Ker}(\mathrm{Weil}(\eta, \overline{\eta}) \to \mathrm{Weil}(X \smallsetminus N, \overline{\eta}))$. We want to prove that the action of K^I on $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is trivial. Let x be a closed point of $X \smallsetminus N$. Since for every $i \in I$, the restriction $\mathcal{H}_{I,W}\big|_{(X \smallsetminus N) \times (x)^{I - \{i\}}}$ is ind-lisse, we deduce that the action of K (the i-th factor in K^I) on $\mathcal{H}_{I,W}\big|_{\overline{\eta} \times (\overline{x})^{I - \{i\}}}$ is trivial. Since $\mathcal{H}_{I,W}$ is ind-lisse over $(X \smallsetminus N)^I$, we have an equivariant isomorphism $\mathcal{H}_{I,W}\big|_{\overline{\eta} \times (\overline{x})^{I - \{i\}}} \xrightarrow{\sim} \mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$. Considering this for every $i \in I$, we deduce that the action of K^I on $\mathcal{H}_{I,W}\big|_{\overline{\eta_I}}$ is trivial. □

Remark 3.2.7. *For any geometric point* $\overline{v}$ *of* $X \smallsetminus N$*, let* $S = (X \smallsetminus N)_{(\overline{v})}$ *be the strict henselization of* $X \smallsetminus N$ *at* $\overline{v}$ *and let* S^+ *be the normalization of* $\overline{\eta}$ *in* S*. By Theorem 3.2.3,* $\mathcal{H}_{I,W}$ *is ind-lisse over* $(X \smallsetminus N)^I$*, so its pullback is ind-lisse over* $(S^+)^I$*. By Proposition 2.3.3,* $\mathcal{H}_{I,W}\big|_{(\overline{\eta})^I}$ *is a constant sheaf over* $(\overline{\eta})^I$*. Since* $(\overline{\eta})^I$ *is open in* $(S^+)^I$*, we deduce that* $\mathcal{H}_{I,W}$ *is constant over* $(S^+)^I$*.*

3.3. **Smoothness of $\mathcal{H}_{I,W}^{\leq \mu}$.** The statement as well as the proof of the following theorem are due to Dennis Gaitsgory and Yakov Varshavsky. (Private communication.)

In this subsection, we revert to the original notation in the introduction and denote by $\mathcal{H}_{G,N,I,W}^j$ the degree j cohomoplogy sheaf and $\mathcal{H}_{G,N,I,W}$ the complex of cohomoplogy sheaves.

Theorem 3.3.1. *For* $\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^+$ *sufficiently regular (i.e. far away from every wall in the Weyl chamber),*

(1) $\mathcal{H}_{G,N,I,W}^{j, \leq \mu}$ *is lisse over* $(X \smallsetminus N)^I$*.*

(2) the morphism $\mathcal{H}_{G,N,I,W}^{j, \leq \mu} \to \mathcal{H}_{G,N,I,W}^j$ *is injective.*

3.3.2. The proof uses Theorem 3.2.3 and the constant term morphisms constructed in [Xue20a] (for $\Lambda = E$) and [Xue20c] (for $\Lambda = \mathcal{O}_E, k_E$). Let's briefly recall:

Let P be a parabolic subgroup of G and M be its Levi quotient. For $W \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, we can view W as a representation of $\widehat{M}^I$ via the canonical inclusion $\widehat{M}^I \hookrightarrow \widehat{G}^I$. Then we define the complex of cohomology sheaves of stack of M-shtukas $\mathcal{H}'_{M,N,I,W}$.

In [Xue20a, Section 4.1], for any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we defined a set $S_M(\mu) := \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} \mid \mu - \lambda$ is a linear combination of simple coroots of M with coefficients in $\mathbb{Q}_{\geq 0}$ modulo $\widehat{\Lambda}^{\mathbb{Q}}_{Z_G}\}$. We define

$$\mathrm{Cht}^{S_M(\mu)}_{G,N,I,W} = \bigcup_{\lambda \in S_M(\mu)} \mathrm{Cht}^{=\lambda}_{G,N,I,W}$$

where $\mathrm{Cht}^{=\lambda}_{G,N,I,W}$ is the inverse image of $\mathrm{Bun}_G^{=\lambda}$ (defined in *loc.cit.* Definition 4.1.3) by $\mathrm{Cht}_{G,N,I,W} \to \mathrm{Bun}_G$, the morphism which sending a G-shtuka to the corresponding G-bundle. We define

$$\mathcal{H}^{S_M(\mu)}_{G,N,I,W} := \mathfrak{p}_!(\mathcal{F}_{G,N,I,W}|_{\mathrm{Cht}^{S_M(\mu)}_{G,N,I,W}/\Xi}).$$

Similarly we define $\mathrm{Cht}'^{\,S_M(\mu)}_{M,N,I,W}$ and $\mathcal{H}'^{\,S_M(\mu)}_{M,N,I,W}$.

Then *loc.cit.* Proposition 4.6.4 (the construction is geometric, so works for any coefficients) says that for $\mu \in \widehat{\Lambda}^+_{G^{\mathrm{ad}}}$ sufficiently regular, the truncated constant term morphism over $(X \smallsetminus N)^I$:

$$\mathcal{C}_G^{P,S_M(\mu)} : \mathcal{H}^{S_M(\mu)}_{G,N,I,W} \to \mathcal{H}'^{\,S_M(\mu)}_{M,N,I,W} \tag{3.15}$$

is an isomorphism.

Now we are ready to give the proof of Theorem 3.3.1. We say that a complex is (ind-)lisse if every cohomology is (ind-)lisse.

Proof. In the proof, we denote $\mathcal{H}_G := \mathcal{H}_{G,N,I,W}$ and $\mathcal{H}_M := \mathcal{H}'_{M,N,I,W}$.

(1) We use an induction argument on the semisimple rank of G. When G is of semisimple rank 0, i.e. G is a torus, there is only one element in $\widehat{\Lambda}^+_{G^{\mathrm{ad}}}$, so only one term in the inductive limit $\mathcal{H}_G$. It is constructible and by Theorem 3.2.3 it is lisse.

Now we suppose that Theorem 3.3.1 (1) is true for every proper Levi subgroup M. For any $\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we have the following fact: $\mathcal{H}_M^{S_M(\lambda)}$ equals to $\mathcal{H}_M^{\leq \lambda'}$ for some dominant co-weight λ' of M. (We refer to [Xue20a, Section 4.1 and the proof of Lemma 5.3.4] for details.) For λ sufficiently regular, λ' is also sufficiently regular, by the induction hypothesis, $\mathcal{H}_M^{\leq \lambda'}$ is lisse over X^I, so is $\mathcal{H}_M^{S_M(\lambda)}$. By (3.15), $\mathcal{H}_G^{S_M(\lambda)}$ is also lisse over X^I.

For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, the open immersion $\mathrm{Cht}^{\leq \mu}_{G,I,W} \to \mathrm{Cht}_{G,I,W}$ induces a cone in the derived category (∞-derived category of ind-constructible sheaves):

$$\mathcal{H}_G^{\leq \mu} \to \mathcal{H}_G \to \mathcal{H}_G / \mathcal{H}_G^{\leq \mu} \xrightarrow{+} \tag{3.16}$$

Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ be sufficiently regular. We construct a zigzag chain

$$\mu = \lambda_0 < \lambda_1 < \cdots < \lambda_i < \cdots$$

as in the proof of [Xue20a, Lemma 5.3.6] and [Xue20b, Proposition 2.2.4] (but in the inverse sense). To construct the chain, for $i \geq 1$, choose $\lambda_i = \lambda_{i-1} + \frac{1}{r}\beta_i$ for some simple coroot β_i of G such that $\lambda_i \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ is still sufficiently regular (here we introduce $r \in \mathbb{N}$ only for technical reason. We fix r as in [Xue20a, 5.1.1]). Take M_i to be the Levi subgroup whose simple coroots are the simple coroots of G except β_i. By definition we have

$$\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}, \lambda \leq \lambda_i\} - \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}, \lambda \leq \lambda_{i-1}\} = S_{M_i}(\lambda_i).$$

An example of the chain is given in Example 3.3.3 below.

For every λ_i ($i \geq 1$) in the chain we have a cone in the derived category

$$\mathcal{H}_G/\mathcal{H}_G^{\leq \lambda_i} \to \mathcal{H}_G/\mathcal{H}_G^{\leq \lambda_{i-1}} \to \mathcal{H}_G^{\leq \lambda_i}/\mathcal{H}_G^{\leq \lambda_{i-1}} \xrightarrow{+}$$

and $\mathcal{H}_G^{\leq \lambda_i}/\mathcal{H}_G^{\leq \lambda_{i-1}} \cong \mathcal{H}_G^{S_{M_i}(\lambda_i)}$. In this way the quotient $\mathcal{H}_G/\mathcal{H}_G^{\leq \mu}$ has a filtration with associated graded $\mathcal{H}_G^{S_{M_i}(\lambda_i)}$.

Since all λ_i are sufficiently regular, $\mathcal{H}_G^{S_{M_i}(\lambda_i)}$ are lisse. Taking into account Lemma A.0.3, we deduce that $\mathcal{H}_G/\mathcal{H}_G^{\leq \mu}$ is ind-lisse over X^I.

By Theorem 3.2.3, $\mathcal{H}_G$ is ind-lisse over X^I. We deduce from (3.16) that $\mathcal{H}_G^{\leq \mu}$ is lisse over X^I. In particular, for any degree $j \in \mathbb{Z}$, $\mathcal{H}_G^{j, \leq \mu}$ is lisse. □

(2) In [Xue20a, Proposition 5.1.5 (c)] and [Xue20c, Proposition 3.2.7 (c)], we proved that for μ sufficiently regular, the restriction of the morphism $\mathcal{H}_G^{j,\leq \mu} \to \mathcal{H}_G^j$ over $\overline{\eta_I}$ is injective (this is a consequence of the following fact, which is proved in [Xue20a, Proposition 5.1.5 (b)] and [Xue20c, Proposition 3.2.7 (b)]: there exists μ_0 sufficiently regular, such that for any $\lambda \geq \mu_0$ and λ sufficiently regular, the morphism

$$\mathrm{Ker}(H_G^{j,\leq \mu_0} \xrightarrow{\mathrm{CT}} \prod_{P \subsetneq G} H'^j_M) \to \mathrm{Ker}(H_G^{j,\leq \lambda} \xrightarrow{\mathrm{CT}} \prod_{P \subsetneq G} H'^j_M)$$

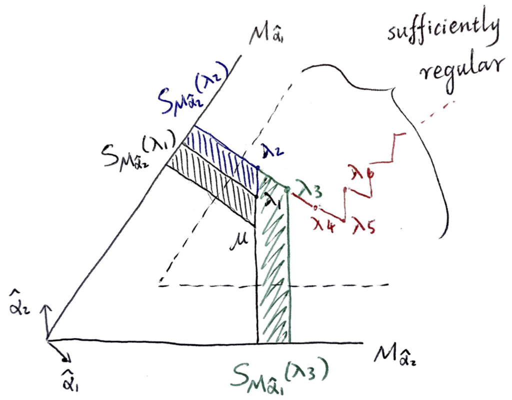
is surjective, where $H_G^j = \mathcal{H}_G^j\big|_{\overline{\eta_I}}$, $H_M^j = \mathcal{H}_M^j\big|_{\overline{\eta_I}}$ and CT are the constant term morphisms. Besides, for λ large enough, we have $\mathrm{Ker}(H_G^{j,\leq \mu_0} \to H_G^j) = \mathrm{Ker}(H_G^{j,\leq \mu_0} \to H_G^{j,\leq \lambda})$. We deduce from these two facts that for λ sufficiently regular, $H_G^{j,\leq \lambda} \to H_G^j$ is injective). Since both sheaves are (ind)-lisse, the morphism itself is injective. □

Example 3.3.3. (of the chain.) Let $G = GL_3$. It has two simple coroots: $\widehat{\alpha}_1$, $\widehat{\alpha}_2$. Let $M_{\widehat{\alpha}_1} = GL_1 \times GL_2$, whose simple coroots are the simple coroots of G except $\widehat{\alpha}_1$, and $M_{\widehat{\alpha}_2} = GL_2 \times GL_1$, whose simple coroots are the simple coroots of G except $\widehat{\alpha}_2$.

In the following picture we illustrate a chain $\mu = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots$. In this chain, $\lambda_1 = \mu + \frac{1}{r}\widehat{\alpha}_2$, i.e. using the notation in the proof of Theorem 3.3.1, we take $\beta_1 = \widehat{\alpha}_2$ and $M_1 = M_{\widehat{\alpha}_2}$. We have

$$\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}, \lambda \leq \lambda_1\} - \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}, \lambda \leq \lambda_0\} = S_{M_{\widehat{\alpha}_2}}(\lambda_1).$$

Similarly $\lambda_2 = \lambda_1 + \frac{1}{r}\widehat{\alpha}_2$ (where $\beta_2 = \widehat{\alpha}_2$ and $M_2 = M_{\widehat{\alpha}_2}$). If we continue in the direction of $\widehat{\alpha}_2$ we will be out of the zone "sufficiently regular", so we change to the direction of $\widehat{\alpha}_1$ and let $\lambda_3 = \lambda_2 + \frac{1}{r}\widehat{\alpha}_1$ (where $\beta_3 = \widehat{\alpha}_1$ and $M_3 = M_{\widehat{\alpha}_1}$). Then we continue...



Remark 3.3.4. *We do not know a direct proof of Theorem 3.3.1 without using Theorem 3.2.3.*

Since we have proved Theorem 3.3.1, we can finally write $\mathcal{H}_{I,W}$ as the inductive limit of lisse sheaves $\mathcal{H}_{I,W}^{\leq\mu}$.

4. The case of non necessarily split groups

Now let G be a geometrically connected smooth reductive group over F, i.e. over the generic point η of X. As in [Laf18, Section 12], let U be the maximal open subscheme of X such that G extends to a smooth reductive group scheme over U. We choose a parahoric integral model of G at all points of $X \smallsetminus U$. Gluing these integral models over U and over the formal neighborhoods of the points of $X \smallsetminus U$, we obtain a smooth group scheme over X. We still denote it by G.

Let ${}^L G$ be the L-group over Λ.

We denote by $\widehat{N} = |N| \cup (X \smallsetminus U)$. We use the definition of cohomology sheaves of stacks of shtukas in [Laf18, Section 12], where we use the geometric Satake equivalence ([Zhu15] for $\Lambda = E$, [ALRR24] for $\Lambda = E, \mathcal{O}_E, k_E$, the properties that we needed are stated in [Laf18, Theoreme 12.16]). For any finite set I and any W finite type Λ-linear representation of $({}^L G)^I$ we have the complex of cohomology sheaves

$$\mathcal{H}_{G,N,I,W} := \varinjlim_{\mu} \mathcal{H}_{G,N,I,W}^{\leq\mu}$$

over $(X \smallsetminus \widehat{N})^I$, where the Harder-Narasimhan truncations are given in [Laf18, Section 12].

It is equipped with an action of the partial Frobenius morphisms and an action of the Hecke algebra.

Sections 1-3 (except 3.3) still work if we replace everywhere $(X \smallsetminus N)^I$ by $(X \smallsetminus \widehat{N})^I$ and $\widehat{G}$ by ${}^L G$.

Remark 4.0.1. *Theorem 3.3.1 should still hold, once we have the constant term morphisms for general reductive groups. However, the constant term morphisms are only written down for split groups for the moment. For non split groups, the construction will need the generalization of the Harder-Narasimhan stratification of* [Sch15] *to non split groups, which are not yet written down (we do not think there are really difficulty).*

Compare with the easy Harder-Narasimhan stratification given by GL_n, the Harder-Narasimhan stratification in [Sch15] *is more canonical and really related to parabolic induction. For Eichler-Shimura relation, the former one is enough. But to construct the constant term morphisms, it would be better to use the latter one.*

Appendix A. A reminder on ind-lisse sheaves

A.0.1. We use [SGA4] VIII 7 for the definition of specialization maps.

Let Y be a normal irreducible noetherian scheme over $\mathbb{F}_q$. By [SGA4] IX Proposition 2.11, a constructible Λ-sheaf $\mathcal{F}$ over Y is lisse if and only if for any geometric points $\overline{x}$, $\overline{y}$ of Y and any specialization map $\mathfrak{sp} : \overline{y} \to \overline{x}$, the induced morphism

$$\mathfrak{sp}^* : \mathcal{F}\big|_{\overline{x}} \to \mathcal{F}\big|_{\overline{y}}$$

is an isomorphism.

A.0.2. Let $\mathcal{H} = \varinjlim_{\lambda \in \Omega} \mathcal{F}_\lambda$ be an inductive limit of constructible Λ-sheaves over a scheme Y, where Ω is a filtered set. We say that the ind-constructible Λ-sheaf $\mathcal{H}$ is *ind-lisse* if we can write $\mathcal{H}$ as an inductive limit of lisse Λ-sheaves over Y, i.e. there exists a filtered set Ω' and lisse Λ-sheaves $\mathcal{G}_{\lambda'}$ for $\lambda' \in \Omega'$ such that $\mathcal{H} \simeq \varinjlim_{\lambda' \in \Omega'} \mathcal{G}_{\lambda'}$.

Lemma A.0.3. *Let Y be a normal irreducible noetherian scheme over $\mathbb{F}_q$. An ind-constructible Λ-sheaf $\mathcal{H}$ over Y is ind-lisse if and only if for any geometric points $\overline{x}$, $\overline{y}$ of Y and any specialization map $\mathfrak{sp} : \overline{y} \to \overline{x}$, the induced morphism*

$$\mathfrak{sp}^* : \mathcal{H}\big|_{\overline{x}} \to \mathcal{H}\big|_{\overline{y}}$$

is an isomorphism.

A.0.4. To prove Lemma A.0.3, we need some preparations. Let $\mathcal{H} = \varinjlim_{\lambda \in \Omega} \mathcal{F}_\lambda$ as above. For any $\lambda \leq \mu$ in Ω, the kernel $\mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_\mu)$ is a constructible sub-Λ-sheaf of $\mathcal{F}_\lambda$. For $\lambda \leq \mu_1 \leq \mu_2$, we have

$$\mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_1}) \subset \mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_2}) \subset \mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{H}) \subset \mathcal{F}_\lambda.$$

Since $\mathcal{F}_\lambda$ is constructible and Y is noetherian, we deduce that there exists λ_0, such that for all $\mu \geq \lambda_0$, we have $\mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\lambda_0}) = \mathrm{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_\mu)$. (The argument is similar to the proof of Lemma 58.73.2 of [StacksProject].) So $\mathrm{Im}(\mathcal{F}_\lambda \to \mathcal{F}_{\lambda_0}) \xrightarrow{\sim} \mathrm{Im}(\mathcal{F}_\lambda \to \mathcal{F}_\mu)$. We denote by $\widetilde{\mathcal{F}_\lambda} := \mathrm{Im}(\mathcal{F}_\lambda \to \mathcal{F}_{\lambda_0})$. We have

$$\varinjlim_{\lambda\in\Omega} \mathcal{F}_\lambda \simeq \varinjlim_{\lambda\in\Omega} \widetilde{\mathcal{F}_\lambda}. \tag{A.1}$$

For any $\lambda_1 \leq \lambda_2$, $\widetilde{\mathcal{F}_{\lambda_1}} \to \widetilde{\mathcal{F}_{\lambda_2}}$ is injective.

Proof of Lemma A.0.3: One direction is obvious. Let's prove the converse direction. Let δ be the generic point of Y and $\overline{\delta}$ a geometric point over δ. By the hypothesis, for any geometric point $\overline{x}$ of Y and any specialization map $\mathfrak{sp} : \overline{\delta} \to \overline{x}$, the induced morphism $\mathfrak{sp}^* : \mathcal{H}\big|_{\overline{x}} \to \mathcal{H}\big|_{\overline{\delta}}$ is an isomorphism.

By A.0.4, we can suppose that for any $\lambda_1 \leq \lambda_2$, $\mathcal{F}_{\lambda_1} \to \mathcal{F}_{\lambda_2}$ is injective. Since every $\mathcal{F}_\lambda$ is a constructible Λ-sheaf over Y, there exists an open dense subscheme U_λ of Y such that $\mathcal{F}_\lambda$ is lisse over U_λ. Let $j_\lambda : U_\lambda \hookrightarrow Y$ be the embedding. Let

$$\mathcal{G}_\lambda := (j_\lambda)_*(\mathcal{F}_\lambda\big|_{U_\lambda}).$$

To prove that $\mathcal{H}$ is ind-lisse, it is enough to prove that

(1) every $\mathcal{G}_\lambda$ is a lisse Λ-sheaf over Y

(2) $\varinjlim_{\lambda\in\Omega} \mathcal{F}_\lambda \simeq \varinjlim_{\lambda\in\Omega} \mathcal{G}_\lambda$

Proof of (1): on the one hand, for every λ, by Lemma A.0.5 below, the morphism $\mathfrak{sp}^* : \mathcal{G}_\lambda\big|_{\overline{x}} \to \mathcal{G}_\lambda\big|_{\overline{\delta}}$ is injective.

On the other hand, for every λ, the adjunction morphism $\mathrm{Id} \to (j_\lambda)_*(j_\lambda)^*$ induces a morphism

$$\mathcal{F}_\lambda \to (j_\lambda)_*(j_\lambda)^*\mathcal{F}_\lambda = \mathcal{G}_\lambda \tag{A.2}$$

Taking limit, we deduce a morphism $\varphi : \varinjlim \mathcal{F}_\lambda \to \varinjlim \mathcal{G}_\lambda$. We have a commutative diagram

$$\begin{array}{ccc} \varinjlim \mathcal{F}_\lambda\big|_{\overline{x}} & \xrightarrow[\simeq]{\mathfrak{sp}^*} & \varinjlim \mathcal{F}_\lambda\big|_{\overline{\delta}} \\ \varphi\downarrow & & \varphi\downarrow\simeq \\ \varinjlim \mathcal{G}_\lambda\big|_{\overline{x}} & \xrightarrow{\mathfrak{sp}^*} & \varinjlim \mathcal{G}_\lambda\big|_{\overline{\delta}} \end{array} \tag{A.3}$$

By the hypothesis the upper line of (A.3) is an isomorphism. By the definition of $\mathcal{G}_\lambda$, the right vertical line of (A.3) is an isomorphism. Thus the lower line of (A.3) is surjective. We want to show that for every λ, $\mathfrak{sp}^* : \mathcal{G}_\lambda\big|_{\overline{x}} \to \mathcal{G}_\lambda\big|_{\overline{\delta}}$ is surjective. Let $a \in \mathcal{G}_\lambda\big|_{\overline{\delta}}$, then there exists $\mu \geq \lambda$ and $b \in \mathcal{G}_\mu\big|_{\overline{x}}$ such that the image of b in $\mathcal{G}_\mu\big|_{\overline{\delta}}$ coincides with the image of a. We identify $\mathcal{G}_\lambda\big|_{\overline{\delta}} = \mathcal{F}_\lambda\big|_{\overline{\delta}}$, $\mathcal{G}_\mu\big|_{\overline{\delta}} = \mathcal{F}_\mu\big|_{\overline{\delta}}$. Since Y is normal irreducible, we identify $\mathcal{G}_\lambda\big|_{\overline{x}} = \Gamma(Y_{(\overline{x})} \times \delta, \mathcal{F}_\lambda)$, $\mathcal{G}_\mu\big|_{\overline{x}} = \Gamma(Y_{(\overline{x})} \times \delta, \mathcal{F}_\mu)$. Since $\mathcal{F}_\lambda\big|_{\overline{\delta}} \subset \mathcal{F}_\mu\big|_{\overline{\delta}}$, if $b \in \Gamma(Y_{(\overline{x})} \times \delta, \mathcal{F}_\mu)$ and the restriction of b to $\mathcal{F}_\mu\big|_{\overline{\delta}}$ is in $\mathcal{F}_\lambda\big|_{\overline{\delta}}$, then b is in $\Gamma(Y_{(\overline{x})} \times \delta, \mathcal{F}_\lambda)$. We deduce that $\mathfrak{sp}^* : \mathcal{G}_\lambda\big|_{\overline{x}} \to \mathcal{G}_\lambda\big|_{\overline{\delta}}$ is surjective.

By A.0.1, we deduce that $\mathcal{G}_\lambda$ is a lisse Λ-sheaf over Y.

Proof of (2): since every $\mathcal{G}_\lambda$ is lisse and for $\lambda_1 \leq \lambda_2$, $\mathcal{G}_{\lambda_1}\big|_\delta \to \mathcal{G}_{\lambda_2}\big|_\delta$ is injective, we deduce that $\mathcal{G}_{\lambda_1} \to \mathcal{G}_{\lambda_2}$ is injective. Since for any λ, the morphism $\mathfrak{sp}^* : \mathcal{G}_\lambda\big|_{\overline{x}} \to \mathcal{G}_\lambda\big|_{\overline{\delta}}$ is injective, we deduce that the lower line of (A.3) is injective. So the lower line of (A.3) is an isomorphism. Thus the left vertical line of (A.3) is also an isomorphism, for any $\overline{x}$.

By Lemma A.0.6 below, $\varphi : \varinjlim \mathcal{F}_\lambda \to \varinjlim \mathcal{G}_\lambda$ is an isomorphism. $\square$

Lemma A.0.5. *Let Y be a normal irreducible noetherian scheme over $\mathbb{F}_q$ and $j : U \hookrightarrow Y$ be an open subscheme. Let $\mathcal{F}$ be a lisse Λ-sheaf over U. Let $\mathcal{G} = j_*\mathcal{F}$. Then for any specialization map $\mathfrak{sp} : \overline{y} \to \overline{x}$, the induced morphism $\mathfrak{sp}^* : \mathcal{G}\big|_{\overline{x}} \to \mathcal{G}\big|_{\overline{y}}$ is injective.*

Proof. It is enough to prove for $\overline{y} = \overline{\delta}$, a geometric generic point of Y. Denote by $Y_{(\overline{x})}$ the strict henselization of Y at $\overline{x}$. Note that $\mathcal{G}\big|_{\overline{x}} = \Gamma(Y_{(\overline{x})}, j_*\mathcal{F}) = \Gamma(Y_{(\overline{x})} \times_Y U, \mathcal{F})$ and $\mathcal{G}\big|_{\overline{\delta}} = \mathcal{F}\big|_{\overline{\delta}}$. By [SGA1] I Proposition 10.1, since Y is normal connected, the fiber product $Y_{(\overline{x})} \times_Y U$ is connected. Since $\mathcal{F}$ is lisse, for any V connected etale over Y, $\Gamma(V, \mathcal{F}) = \mathcal{F}\big|_{\overline{\delta}}^{\pi_1(V,\overline{\delta})}$. We deduce that the restriction

$$\Gamma(Y_{(\overline{x})} \times_Y U, \mathcal{F}) \to \mathcal{F}\big|_{\overline{\delta}}$$

is injective. $\square$

Lemma A.0.6. *Let $\varphi : \varinjlim \mathcal{F}_\lambda \to \varinjlim \mathcal{G}_\lambda$ be a morphism of ind-constructible sheaves over Y induced by $\mathcal{F}_\lambda \to \mathcal{G}_\lambda$ for every λ. If for every geometric point $\overline{y}$, $\varphi\big|_{\overline{y}} : \varinjlim \mathcal{F}_\lambda\big|_{\overline{y}} \to \varinjlim \mathcal{G}_\lambda\big|_{\overline{y}}$ is an isomorphism, then φ is an isomorphism.*

Proof. By A.0.4, we can suppose that all morphisms in $\varinjlim \mathcal{F}_\lambda$ and in $\varinjlim \mathcal{G}_\lambda$ are injective. For any λ and any $\overline{y}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_\lambda\big|_{\overline{y}} & \hookrightarrow & \varinjlim \mathcal{F}_\lambda\big|_{\overline{y}} \\ \downarrow & & \downarrow \simeq \\ \mathcal{G}_\lambda\big|_{\overline{y}} & \hookrightarrow & \varinjlim \mathcal{G}_\lambda\big|_{\overline{y}} \end{array} \tag{A.4}$$

We deduce that $\mathcal{F}_\lambda\big|_{\overline{y}} \to \mathcal{G}_\lambda\big|_{\overline{y}}$ is injective. Since this is true for any $\overline{y}$, we deduce that $\mathcal{F}_\lambda \to \mathcal{G}_\lambda$ is injective.

Now fix λ. For any $\mu \geq \lambda$, consider the subset of Y

$$C_\mu := \{y \in Y \text{ such that } \mathcal{G}_\lambda\big|_{\overline{y}} \not\subseteq \mathrm{Im}(\mathcal{F}_\mu\big|_{\overline{y}} \to \mathcal{G}_\mu\big|_{\overline{y}})\}$$

It is constructible. For any $\mu_1 \leq \mu_2$, we have $C_{\mu_1} \supset C_{\mu_2}$. Since $\varphi\big|_{\overline{y}}$ is surjective for any $\overline{y}$, we have $\cap_\mu C_\mu = \emptyset$. We deduce that there exists μ^0 (depending on λ), such that for any $\mu \geq \mu^0$, we have $C_\mu = \emptyset$. In particular, $\mathcal{G}_\lambda \subset \mathrm{Im}(\mathcal{F}_{\mu^0} \to \mathcal{G}_{\mu^0})$. Thus for any λ, we have

$$\mathcal{F}_\lambda \subset \mathcal{G}_\lambda \subset \mathcal{F}_{\mu^0}$$

This implies

$$\varinjlim_{\lambda \in \Omega} \mathcal{F}_\lambda \simeq \varinjlim_{\lambda \in \Omega} \mathcal{G}_\lambda. \tag{A.5}$$

$\square$

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