ON THE QUOTIENT OF A GROUPOID BY AN ACTION OF A 2-GROUP

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To Gérard Laumon with deepest admiration

Abstract. If $\mathscr{X}$ is a groupoid equipped with an action of a 2-group $\mathscr{G}$ then one has a 2-groupoid $\mathscr{X}/\mathscr{G}$. We describe the fibers of the functor from $\mathscr{X}/\mathscr{G}$ to the 1-groupoid $\pi_0(\mathscr{X})/\pi_0(\mathscr{G})$. We also give an explicit model for $\mathscr{X}/\mathscr{G}$ in a certain situation.

The work gives an abstract model for a certain 2-stack which provides a conjectural description of the p-adic completion of the stack of n-truncated Barsotti-Tate groups.

If a 2-group $\mathscr{G}$ acts on a groupoid $\mathscr{X}$ then one can form the quotient 2-groupoid $\mathscr{X}/\mathscr{G}$. This general construction is discussed in §1. In §2 we discuss the following special situation: $\mathscr{X}$ is the underlying groupoid of a 2-group, and $\mathscr{G}$ acts on $\mathscr{X}$ by two-sided translations.

This elementary paper is motivated by the following: the sheafified version of the situation of §2 occurs in the definition of the 2-stack from [D, §D.8.3], which provides a conjectural description of the p-adic completion of the stack of n-truncated Barsotti-Tate groups and its "Shimurian" analogs (see Appendix A for more details). The result of §2.4 of this paper could be used to test Conjecture D.8.4 from [D].

I first constructed the 2-groupoid from §2.4 by trial and error (I was motivated by potential applications to the Lau group scheme and the Lau gerbe, see §2.2.2). Then the idea of treating this explicit construction as a special case of the 2-groupoid $\mathscr{X}/\mathscr{G}$ was suggested to me by D. Arinkin and N. Rozenblyum. I am very grateful to them.

1. The quotient 2-groupoid

1.1. **The question.**

1.1.1. *2-groupoids and 2-groups.* Recall that a 2-groupoid is a 2-category in which all 1-morphisms and 2-morphisms are invertible. A 2-*group* is a 2-groupoid with a single object; equivalently, a 2-groupoid is a monoidal category in which all objects and morphisms are invertible.

1.1.2. *Quotient groupoids.* If a group G acts on a set X then the *quotient groupoid* (or groupoid quotient) X/G is defined as follows: the set of objects is X, a morphism $x \to x'$ is an element $g \in G$ such that $gx = x'$, and the composition of morphisms is given by multiplication in G.

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1.1.3. *Quotient 2-groupoids.* More generally, if a 2-group $\mathscr{G}$ acts on a groupoid $\mathscr{X}$ then one defines the *quotient 2-groupoid* $\tilde{\mathscr{X}} = \mathscr{X}/\mathscr{G}$ as follows:

(i) $\operatorname{Ob}\tilde{\mathscr{X}} := \operatorname{Ob}\mathscr{X}$;

(ii) for $x_1, x_2 \in \mathscr{X}$ let $\underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1, x_2)$ be the following groupoid: its objects are pairs

$$(g, f), \text{ where } g \in \mathscr{G}, \quad f \in \operatorname{Isom}(x_2, gx_1),$$

and a morphism $(g, f) \to (g', f')$ is a morphism $g \to g'$ such that the corresponding morphism $gx_1 \to g'x_1$ equals $f'f^{-1}$;

(iii) the composition functor $\underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1, x_2) \times \underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_2, x_3) \to \underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1, x_3)$ comes from the product in $\mathscr{G}$.

1.1.4. *The question.* In the situation of §1.1.3, let $\mathscr{X}' := \pi_0(\mathscr{X})/\pi_0(\mathscr{G})$, where π_0 stands for the set of isomorphism classes of objects and $\pi_0(\mathscr{X})/\pi_0(\mathscr{G})$ is the quotient 1-groupoid in the sense of §1.1.2. We have a canonical functor

$$\tilde{\mathscr{X}} = \mathscr{X}/\mathscr{G} \to \mathscr{X}'. \tag{1.1}$$

It is easy to see that the functor (1.1) induces a surjection at the level of objects and at the level of 1-morphisms. So the obstruction to it being an equivalence is formed by the 2-groups

$$\operatorname{Ker}(\underline{\operatorname{Aut}}_{\tilde{\mathscr{X}}} x \to \operatorname{Aut}_{\mathscr{X}'} \bar{x}), \quad x \in \mathscr{X}, \tag{1.2}$$

where $\bar{x} \in \pi_0(\mathscr{X})$ is the image of x. Let us note that $\operatorname{Aut}_{\mathscr{X}'} \bar{x}$ is just the stabilizer of $\bar{x}$ in $\pi_0(\mathscr{G})$; similarly, $\underline{\operatorname{Aut}}_{\tilde{\mathscr{X}}} x$ is the "categorical stabilizer" of x in $\mathscr{G}$.

The problem is to describe the 2-group (1.2). This will be done in Proposition 1.3.2 in terms of a certain crossed module.

1.2. **Recollections on crossed modules.**

1.2.1. *Crossed modules.* Recall that a *crossed module* $G^\bullet$ is a pair of groups G^0, G^{-1} together with an action of G^0 on G^{-1} and a homomorphism $d : G^{-1} \to G^0$ satisfying certain identities. The image of $\gamma \in G^{-1}$ under the action of $g \in G^0$ is denoted by ${}^g\gamma$, and the identities are as follows:

$$d({}^g\gamma) = gd(\gamma)g^{-1}, \quad \gamma \in G^{-1}, g \in G, \tag{1.3}$$

$${}^{d(\gamma)}\gamma' = \gamma\gamma'\gamma^{-1} \quad \gamma, \gamma' \in G^{-1}. \tag{1.4}$$

1.2.2. *The 2-group corresponding to a crossed module.* A crossed module $G^\bullet$ gives rise to a strict 2-group, which we denote by

$$\operatorname{Cone}(G^{-1} \xrightarrow{d} G^0). \tag{1.5}$$

As a groupoid, this is the quotient of G^0 by the action of G^{-1} given by $(\gamma, g) \mapsto d(\gamma) \cdot g$; thus the set of objects is G^0, and for $g, g' \in G^0$ one has

$$\operatorname{Mor}(g, g') = \{\gamma \in G^{-1} \mid d(\gamma)g = g'\}. \tag{1.6}$$

The tensor product map $\operatorname{Mor}(g_1, g_1') \times \operatorname{Mor}(g_2, g_2') \to \operatorname{Mor}(g_1g_2, g_1'g_2')$ is given by

$$(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot {}^{g_1}\gamma_2. \tag{1.7}$$

The assignment $G^\bullet \mapsto \mathrm{Cone}(G^{-1} \xrightarrow{d} G^0)$ gives an equivalence between the category of crossed modules and that of strict 2-groups (e.g., see [L, Lemma 2.2]).

1.2.3. *The abelian case.* We say that a crossed module $G^\bullet$ is *abelian* if G^0, G^{-1} are abelian and the action of G^0 on G^{-1} is trivial. In this case the 2-group from §1.2.2 is the one associated in [SGA4, Exposé XVIII, §1.4] to the complex

$$0 \to G^{-1} \xrightarrow{d} G^0 \to 0, \tag{1.8}$$

so there is no conflict between understanding $\mathrm{Cone}(G^{-1} \xrightarrow{d} G^0)$ as a 2-group and the usual understanding as the complex (1.8).

1.3. Answering the question from §1.1.4.

1.3.1. *A crossed module related to the $\mathscr{G}$-action.* We keep the notation of §1.1.3-1.1.4. Let $\pi_1(\mathscr{G}) := \mathrm{Aut}(1_{\mathscr{G}})$, where $1_{\mathscr{G}}$ is the unit object of the 2-group $\mathscr{G}$; it is well known that $\pi_1(\mathscr{G})$ is abelian. The $\mathscr{G}$-action induces for each $x \in \mathscr{X}$ a homomorphism

$$\phi_x : \pi_1(\mathscr{G}) \to \mathrm{Aut}_{\mathscr{X}}\, x, \tag{1.9}$$

which is functorial in x, i.e., for any $\psi \in \mathrm{Mor}(x,y)$ one has $\psi \circ \phi_x = \phi_y \circ \psi$. Applying this for $y = x$, we see that $\mathrm{Im}\, \phi_x$ *is contained in the center of* $\mathrm{Aut}_{\mathscr{X}}\, x$. So we can regard (1.9) as a crossed module in which the action of $\mathrm{Aut}_{\mathscr{X}}\, x$ on $\pi_1(\mathscr{G})$ is trivial. Let $\mathrm{Cone}(\phi_x)$ be the corresponding 2-group (see §1.2.2).

Proposition 1.3.2. *There is a canonical equivalence of 2-groups*

$$\mathrm{Cone}(\pi_1(\mathscr{G}) \xrightarrow{\phi_x} \mathrm{Aut}_{\mathscr{X}}) \xrightarrow{\sim} \mathrm{Ker}(\underline{\mathrm{Aut}}_{\tilde{\mathscr{X}}} x \to \mathrm{Aut}_{\mathscr{X}'}\, \bar{x}). \tag{1.10}$$

Proof. By §1.1.3, objects of $\underline{\mathrm{Aut}}_{\tilde{\mathscr{X}}} x$ are pairs (g, f), where $g \in \mathscr{G}$ and $f : x \xrightarrow{\sim} gx$. A morphism $(g,f) \to (g',f')$ is a morphism $g \to g'$ inducing $f' \circ f^{-1} : gx \to g'x$. The product in $\underline{\mathrm{Aut}}_{\tilde{\mathscr{X}}} x$ is

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1 g_2, \tilde{f}_2 \circ f_1),$$

where $\tilde{f}_2 : g_1 x \to g_1 g_2 x$ comes from $f_2 : x \to g_2 x$.

The equivalence (1.10) takes $f \in \mathrm{Aut}_{\mathscr{X}}\, x$ to the pair $(1_{\mathscr{G}}, f) \in \underline{\mathrm{Aut}}_{\tilde{\mathscr{X}}} x$; at the level of morphisms, it comes from $\mathrm{id} : \pi_1(\mathscr{G}) \to \pi_1(\mathscr{G})$. □

1.4. **Some corollaries.** Let $\tilde{\mathscr{X}}^{\le 1}$ be the 1-truncation of $\tilde{\mathscr{X}}$, i.e., the 1-groupoid obtained by replacing the groupoids $\underline{\mathrm{Mor}}_{\tilde{\mathscr{X}}}(x_1, x_2)$, $x_i \in \tilde{\mathscr{X}}$, by their π_0's. The functor $\tilde{\mathscr{X}} \to \mathscr{X}'$ factors as $\tilde{\mathscr{X}} \to \tilde{\mathscr{X}}^{\le 1} \to \mathscr{X}'$. The functor

$$\tilde{\mathscr{X}}^{\le 1} \to \mathscr{X}' \tag{1.11}$$

is a gerbe.[1]

Corollary 1.4.1. *For each $x \in \mathscr{X}$, one has*

(i) $\mathrm{Ker}(\mathrm{Aut}_{\tilde{\mathscr{X}}^{\le 1}}(x) \to \mathrm{Aut}_{\mathscr{X}'}(x)) = \mathrm{Coker}\, \phi_x$.

(ii) $\mathrm{Aut}_{\mathscr{X}}(\mathrm{id}_x) = \mathrm{Ker}\, f_x$.

[1] A functor between groupoids is said to be a gerbe if each of its fibers has one and only one isomorphism class of objects.

1.4.2. *The abelian situation.* Suppose that for each $x \in \mathscr{X}$ the group $\pi_1(\mathscr{X}, x) = \operatorname{Aut}_{\mathscr{X}} x$ is abelian. Then $\pi_1(\mathscr{X}, x)$ and the homomorphism (1.9) depend only on $\bar{x} \in \pi_0(\mathscr{X})$, so we can rewrite (1.9) as

$$\phi_{\bar{x}} : \pi_1(\mathscr{G}) \to \pi_1(\mathscr{X}, \bar{x}), \quad \bar{x} \in \pi_0(\mathscr{X}). \tag{1.12}$$

Let $\mathfrak{L}(\bar{x}) := \operatorname{Coker} \phi_{\bar{x}}$.

The group $\pi_0(\mathscr{G})$ acts on $\pi_1(\mathscr{G})$ and $\pi_0(\mathscr{X})$, and the collection of homomorphisms (1.12) is $\pi_0(\mathscr{G})$-equivariant. So the collection of abelian groups $\mathfrak{L}(\bar{x})$, $\bar{x} \in \pi_0(\mathscr{X})$, is $\pi_0(\mathscr{G})$-equivariant. In other words, we get a functor

$$\mathfrak{L} : \mathscr{X}' \to \mathrm{Ab}; \tag{1.13}$$

as before, $\mathscr{X}'$ denotes the quotient groupoid $\pi_0(\mathscr{X})/\pi_0(\mathscr{G})$.

The gerbe (1.11) is banded by the functor $\mathfrak{L}$; this follows from Corollary 1.4.1(i).

2. A particular situation

2.1. **Subject of this section.**

2.1.1. If $\mathfrak{G}$ is a 2-group and $\mathscr{X}$ is the underlying groupoid of $\mathfrak{G}$ then $\mathfrak{G} \times \mathfrak{G}$ acts on $\mathscr{X}$ by two-sided translations: namely, $(g, g') \in \mathfrak{G} \times \mathfrak{G}$ acts by $x \mapsto gx(g')^{-1}$. So given a homomorphism of 2-groups $\mathfrak{B} \to \mathfrak{G} \times \mathfrak{G}$, we get an action of $\mathfrak{B}$ on $\mathscr{X}$ and the corresponding 2-groupoid $\mathscr{X}/\mathfrak{B}$.

2.1.2. *The goal.* Now let $B^\bullet, G^\bullet$ be crossed modules and $\pi, \pi' : B^\bullet \to G^\bullet$ be homomorphisms. Then we get the homomorphism $(\pi, \pi') : \mathfrak{B} \to \mathfrak{G} \times \mathfrak{G}$, where $\mathfrak{B} := \operatorname{Cone}(B^\bullet)$, $\mathfrak{G} := \operatorname{Cone}(G^\bullet)$. It gives a strict action of the strict 2-group $\mathfrak{B}$ on $\mathscr{X}$ and therefore a strict 2-groupoid $\mathscr{X}/\mathfrak{B}$. We will denote this strict 2-groupoid by

$$\operatorname{Cone}(B^\bullet \xrightarrow{\pi, \pi'} G^\bullet). \tag{2.1}$$

The main goal of §2 is to give a certain tautological reformulation of the construction of the 2-groupoid (2.1). This will be done in §2.4. Let us note two cases in which this reformulation looks nice.

2.1.3. *Two easy cases.* (i) Suppose that $B^\bullet$ and $G^\bullet$ are abelian in the sense of §1.2.3. Then $\operatorname{Cone}(B^\bullet \xrightarrow{\pi - \pi'} G^\bullet)$ is a complex of abelian groups $0 \to C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \to 0$, and $\operatorname{Cone}(B^\bullet \xrightarrow{\pi, \pi'} G^\bullet)$ is the strict 2-group associated to this complex in the usual way: its objects are elements of C^0, and for $c, c' \in C^0$ the groupoid of morphisms $c \to c'$ is the quotient of the set $\{x \in C^{-1} \mid dx = c' - c\}$ by the action of C^{-2}.

(ii) If $B^{-1} = 0$ then $\operatorname{Cone}(B^\bullet \xrightarrow{\pi, \pi'} G^\bullet)$ is a 1-groupoid. To describe it, first note that the maps

$$B^0 \times G^0 \to G^0, \quad (b, g) \mapsto \pi(b) g \pi'(b)^{-1},$$
$$G^{-1} \times G^0 \to G^0, \quad (\gamma, g) \mapsto d(\gamma) g$$

define actions of the groups B^0 and G^{-1} on the set G^0. These actions combine into an action of $B^0 \ltimes_\pi G^{-1}$ on the set G^0, where $B^0 \ltimes_\pi G^{-1}$ is the semidirect product via the homomorphism $B^0 \xrightarrow{\pi} G^0 \to \operatorname{Aut} G^{-1}$. It is easy to check that the corresponding quotient groupoid (in the sense of §1.1.2) is $\operatorname{Cone}(B^\bullet \xrightarrow{\pi, \pi'} G^\bullet)$.

2.1.4. In general, it is easy to check that the objects and 1-morphisms of $\mathrm{Cone}(B^\bullet \xrightarrow{\pi,\pi'} G^\bullet)$ are the same as in §2.1.3(ii). To describe the 2-morphisms, we use a slight generalization of the notion of crossed module, see §2.3 below.

2.2. **The maps ϕ_g.** Let us describe the map (1.9). In our situation, $\mathscr{G} = \mathrm{Cone}(B^\bullet)$ and $\mathscr{X}$ is the underlying groupoid of $\mathrm{Cone}(G^\bullet)$. Note that

$$\pi_1(\mathscr{G}) = H^{-1}(B^\bullet) := \mathrm{Ker}(B^{-1} \xrightarrow{d} B^0).$$

We have $\mathrm{Ob}\,\mathscr{X} = G^0$, and for $g \in G^0$ the group $\mathrm{Aut}_{\mathscr{X}}\, g$ identifies with $H^{-1}(G^\bullet)$ via (1.6) (in particular, $\mathrm{Aut}_{\mathscr{X}}\, g$ is abelian, so we are in the situation of §1.4.2). Thus the homomorphism (1.9) is a map

$$\phi_g : H^{-1}(B^\bullet) \to H^{-1}(G^\bullet).$$

Lemma 2.2.1. *The homomorphism ϕ_g is as follows:*

$$\phi_g(\beta) = \pi(\beta) \cdot {}^g\pi'(\beta)^{-1} = {}^g\pi'(\beta)^{-1} \cdot \pi(\beta), \quad \beta \in H^{-1}(B^\bullet). \tag{2.2}$$

Proof. By (1.7), the map

$$\mathrm{Mor}(g_1, g_1') \times \mathrm{Mor}(g_2, g_2') \times \mathrm{Mor}(g_3, g_3') \to \mathrm{Mor}(g_1g_2g_3, g_1'g_2'g_3')$$

is given by $(\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_1 \cdot {}^{g_1}\gamma_2 \cdot {}^{g_1g_2}\gamma_3$. To get $\phi_g(\beta)$, one has to take $g_1 = g_3 = 1$, $g_2 = g$, $\gamma_1 = \pi(\beta)$, $\gamma_2 = 1$, $\gamma_3 = \pi'(\beta)^{-1}$. □

2.2.2. *Remarks.* (i) Lemma 2.2.1 implies that in our situation the functor $\mathfrak{L}$ from §1.4.2 is $g \mapsto \mathrm{Coker}\,\phi_g$, where ϕ_g is given by (2.2).

(ii) The author hopes that the functor $\mathfrak{L}$ from the previous remark can serve as an abstract model of the Lau group scheme (in the sense of [D, Thm. 1.1.1]) and that the canonical gerbe banded by $\mathfrak{L}$ (see (1.11) and §1.4.2) can serve as an abstract model of the Lau gerbe (by which we mean the gerbe from [D, Thm. 1.1.1]).

2.3. **A way to describe strict 2-groupoids.** A strict 2-groupoid with a single object (a.k.a. a strict 2-group) is "the same as" a crossed module, see §1.2.1-1.2.2. Arbitrary strict 2-groupoids have a similar description via a slight generalization of the notion of crossed module.

2.3.1. *Definition.* Let X be a set. An *X-crossed module* is the following data:

(i) a groupoid Γ with $\mathrm{Ob}\,\Gamma = X$;

(ii) a functor

$$\Gamma \mapsto \{\mathrm{Groups}\}, \quad x \mapsto H_x;$$

(iii) a collection of homomorphisms

$$d_x : H_x \to \mathrm{Aut}_\Gamma\, x$$

such that d_x is functorial in x and

$${}^{d_x(h)}h' = hh'h^{-1} \quad \text{for all } x \in X \text{ and } h, h' \in H_x,$$

where ${}^{d_x(h)}h'$ stands for the image of h' under the automorphism of H_x corresponding to $d_x(h) \in \mathrm{Aut}_\Gamma\, x$ by functoriality of H_x.

If X has a single element one gets the usual notion of crossed module, see §1.2.1.

2.3.2. *From strict 2-groupoids to X-crossed modules.* Let $\mathcal{C}$ be a strict 2-groupoid and $X = \mathrm{Ob}\,\mathcal{C}$. Then one defines an X-crossed module as follows:

(i) Γ is the 1-skeleton of $\mathcal{C}$ (i.e., the 1-groupoid obtained by disregarding the 2-morphisms of $\mathcal{C}$);

(ii) for $x \in X$ one sets $H_x := \mathrm{Ker}(\mathrm{Aut}_\Gamma\, x \twoheadrightarrow \underline{\mathrm{Aut}}_{\mathcal{C}} x)$, where Ker stands for the categorical fiber over $\mathrm{id}_x \in \underline{\mathrm{Aut}}_{\mathcal{C}} x$; so H_x is formed by pairs (g, f), where $g \in \mathrm{Aut}_\Gamma\, x$ and $f : g \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}\, x$ is a 2-morphism in $\mathcal{C}$;

(iii) the map $d_x : H_x \to \mathrm{Aut}_\Gamma\, x$ forgets f.

We have defined a functor from the 1-category of strict 2-groupoids to the category of pairs consisting of a set X and an X-crossed module. This functor is an equivalence, and the inverse functor is described below.

2.3.3. *From X-crossed modules to strict 2-groupoids.* In the situation of §2.3.1 we have to define the groupoids $\underline{\mathrm{Mor}}_{\mathcal{C}}(x, x')$ for $x, x' \in X$ and the composition functors

$$\underline{\mathrm{Mor}}(x', x'') \times \underline{\mathrm{Mor}}(x, x') \to \underline{\mathrm{Mor}}(x, x''). \tag{2.3}$$

for $x, x', x'' \in X$.

$\underline{\mathrm{Mor}}_{\mathcal{C}}(x, x')$ is defined to be the groupoid quotient of the set $\mathrm{Mor}_\Gamma(x, x')$ with respect to the action of $H_{x'}$ that comes from the homomorphism $d_{x'} : H_x \to \mathrm{Aut}_\Gamma\, x'$. Thus a morphism in $\underline{\mathrm{Mor}}(x, x')$ is a triple

$$\alpha = (g, \tilde{g}, h),$$

where $g, \tilde{g} \in \mathrm{Mor}(x, x')$ are the source and target of α, and $h \in H_{x'}$ is such that $\tilde{g} = d_{x'}(h) \cdot g$.

At the level of objects, the functor (2.3) is just the composition map in the groupoid Γ. Let us define (2.3) at the level of morphisms. Let α_1 (resp. α_2) be a morphism in $\underline{\mathrm{Mor}}(x', x'')$ (resp. $\underline{\mathrm{Mor}}(x, x')$). As above, write α_i as a triple $(g_i, \tilde{g}_i, h_i)$. The image of (α_1, α_2) under (2.3) is defined to be the triple

$$(g_1 g_2,\ \tilde{g}_1 \tilde{g}_2,\ h_1 \cdot {}^{g_1} h_2)$$

similarly to formula (1.7).

2.4. **Describing the 2-groupoid (2.1).** Let X be the underlying set of G^0. Let us describe the X-crossed module corresponding to the strict 2-groupoid $\mathrm{Cone}(B^\bullet \xrightarrow{\pi, \pi'} G^\bullet)$ (the 2-groupoid itself can be recovered from the X-crossed module as explained in §2.3.3).

The 1-groupoid Γ is the one corresponding to the action of the group $B^0 \ltimes_\pi G^{-1}$ on X described in §2.1.3(ii). It remains for us to describe the data from §2.3.1(ii-iii), i.e., the $(B^0 \ltimes_\pi G^{-1})$-equivariant family of groups H_g, $g \in G^0$, and the homomorphisms

$$d_g : H_g \to \mathrm{Stab}_g, \quad g \in G^0,$$

where $\mathrm{Stab}_g \subset B^0 \ltimes_\pi G^{-1}$ is the stabilizer of g, i.e.,

$$\mathrm{Stab}_g = \{ b \cdot \gamma \mid b \in B^0,\ \gamma \in G^{-1},\ d(\gamma) = \pi(b)^{-1} \cdot g \pi'(b) g^{-1} \}. \tag{2.4}$$

It is easy to check that these data are as follows:

(i) $H_g = B^{-1}$, and the $B^0 \ltimes_\pi G^{-1}$-equivariant structure comes from the action of B^0 on B^{-1};

(ii) in terms of (2.4), the map $d_g : B^{-1} \to \mathrm{Stab}_g$ is given by $b = d(\beta)$, $\gamma = \pi(\beta)^{-1} \cdot {}^g\pi'(\beta)$, where $\beta \in B^{-1}$; so

$$d_g(\beta) = d(\beta) \cdot \pi(\beta)^{-1} \cdot {}^g\pi'(\beta), \quad \beta \in B^{-1}. \tag{2.5}$$

2.4.1. *Remark.* It is easy to check that the map

$$d_g : B^{-1} \to B^0 \ltimes_\pi G^{-1}, \quad \beta \mapsto d(\beta) \cdot \pi(\beta)^{-1}$$

is a homomorphism whose image centralizes G^{-1}. So the map (2.5) is a homomorphism. Moreover, one could rewrite (2.5) as $d_g(\beta) = {}^g\pi'(\beta) \cdot d(\beta) \cdot \pi(\beta)^{-1}$.

Appendix A. Relation to the 2-stack $\mathrm{BT}_n^{G,\mu,?}$

A.1. **Goal of this Appendix.** Let $\mathbb{B}^\bullet, \mathbb{G}^\bullet$ be crossed modules in some topos. Suppose we are given homomorphisms $\pi, \pi' : \mathbb{B}^\bullet \to \mathbb{G}^\bullet$. Then one defines a 2-stack

$$\mathrm{Cone}(\mathbb{B}^\bullet \xrightarrow{\pi,\pi'} \mathbb{G}^\bullet) \tag{A.1}$$

similarly to §2.1.2; if the topos is a point then (A.1) is the 2-groupoid (2.1).

Under a very mild assumption[2], the 2-stack $\mathrm{BT}_n^{G,\mu,?}$ from [D, §D.8.3] can be writtten in the form (A.1) in a rather natural way. The goal of this Appendix is to provide some details about this in a somewhat informal way.

A.2. **The topos.** Throughout this Appendix, we fix a prime p. A ring R is said to be p-nilpotent if the element $p \in R$ is nilpotent. Let p-Nilp denote the category of p-nilpotent rings. The topos relevant for us is the category of fpqc-sheaves of sets on $\text{p-Nilp}^{\text{op}}$. From now on, the word "stack" will refer to this topos.

A.3. **The stack $\mathrm{BT}_n^{G,\mu}$.**

A.3.1. *The results of* [GM]. Let $n \in \mathbb{N}$. Let G be a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and

$$\mu : \mathbb{G}_m \to G$$

a cocharacter.

For $R \in$ p-Nilp let $\mathrm{BT}_n^{G,\mu}(R)$ be as in [GM, §9]; this is the ∞-groupoid[3] of G-bundles on $R^{\mathrm{Syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ satisfying a certain condition, which depends on μ. Here R^{Syn} is the syntomification of R.

μ is said to be 1-*bounded* if all weights of the action of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1 (if G is reductive and almost simple this means that μ is minuscule or zero). By [GM, Thm. D], if μ is 1-bounded then $\mathrm{BT}_n^{G,\mu}$ is a smooth algebraic stack over $\mathrm{Spf}\,\mathbb{Z}_p$; in other words, for every $m \in \mathbb{N}$ the restriction of $\mathrm{BT}_n^{G,\mu}$ to the category of $\mathbb{Z}/p^m\mathbb{Z}$-algebras is a smooth algebraic 1-stack over $\mathbb{Z}/p^m\mathbb{Z}$. By [GM, Thm. A], if $G = GL(d)$ and μ is 1-bounded then $\mathrm{BT}_n^{G,\mu}$ identifies with the stack of n-truncated Barsotti-Tate groups of height d and dimension d', where d' depends on μ.

A.3.2. *The 2-stack* $\mathrm{BT}_n^{G,\mu,?}$ *and the conjecture.* §D.8.3 of [D] contains a definition of a certain 2-stack $\mathrm{BT}_n^{G,\mu,?}$; in this definition μ is not required to be 1-bounded. Conjecture D.8.4 from [D] says that if μ is 1-bounded then $\mathrm{BT}_n^{G,\mu} = \mathrm{BT}_n^{G,\mu,?}$ (which implies that $\mathrm{BT}_n^{G,\mu,?}$ is a 1-stack if μ is 1-bounded).

[2] See §A.3.3 below.

[3] If R is good enough (e.g., l.c.i) then the derived stack $R^{\mathrm{Syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ is classical, so $\mathrm{BT}_n^{G,\mu}(R)$ is a 1-groupoid.

A.3.3. *To be explained below.* Assume that G is lifted to a smooth affine group scheme over $\mathbb{Z}_p$ (note that such a lift is automatic if G is reductive). Our goal is to explain why $\mathrm{BT}_n^{G,\mu,?}$ can be rather naturally written in the form (A.1), where $\mathbb{B}^i$ and $\mathbb{G}^i$ are explicit group ind-schemes over $\mathbb{Z}_p$.

A.4. **Format of the definition of** $\mathrm{BT}_n^{G,\mu,?}$**.**

A.4.1. *The story in a few words.* According to the definition from [D, §D.8.3], the 2-stack $\mathrm{BT}_n^{G,\mu,?}$ is obtained by applying the construction of §2.1.1 to a certain homomorphism of group stacks

$$\mathfrak{B} \to \mathfrak{G} \times \mathfrak{G}, \tag{A.2}$$

see formula (D.10) of [D]. It turns out that this homomorphism lifts in a rather natural way to a homomorphism

$$\mathbb{B}^\bullet \xrightarrow{(\pi,\pi')} \mathbb{G}^\bullet \times \mathbb{G}^\bullet$$

of fpqc sheaves of crossed modules. Such a lift provides a realization of $\mathrm{BT}_n^{G,\mu,?}$ in the form (A.1).

A.4.2. *What will be explained.* Instead of discussing diagram (A.2), we will only discuss $\mathfrak{G}$: we will recall the definition of the 2-stack $\mathfrak{G}$ from [D] and explain how to lift it to a sheaf of crossed modules $\mathbb{G}^\bullet$. Let us note that $\mathfrak{G}$ does not depend on the cocharacter μ (but $\mathfrak{B}$ does).

A.4.3. *The 2-stack* $\mathfrak{G}$*.* By definition,

$$\mathfrak{G} := G({}^s\mathscr{R}_n), \tag{A.3}$$

where ${}^s\mathscr{R}_n$ is a certain stack of $\mathbb{Z}/p^n\mathbb{Z}$-algebras, whose definition is sketched[4] in [D, §D.7.1]. Formula (A.3) just means that

$$\mathfrak{G}(A) := G({}^s\mathscr{R}_n(A)), \quad A \in \text{p-Nilp}\,. \tag{A.4}$$

The r.h.s of (A.4) makes sense because G is a scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and ${}^s\mathscr{R}_n(A)$ is an animated $\mathbb{Z}/p^n\mathbb{Z}$-algebra.

A.5. **Constructing the crossed module** $\mathbb{G}^\bullet$**.**

A.5.1. *Recollections.* In addition to §1.2.1 and the interpretation via strict 2-groups at the end of §1.2.2, there are two other well known points of view on crossed modules:

(i) a crossed module is the same as a 2-group $\mathfrak{G}$ with an epimorphism $G^0 \twoheadrightarrow \mathfrak{G}$, where G^0 is a group;

(ii) a crossed module is the same as a groupoid[5] internal to the category of groups.

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[4] The stack ${}^s\mathscr{R}_{n,\mathbb{F}_p} := {}^s\mathscr{R}_n \times \operatorname{Spec} \mathbb{F}_p$ is *completely described* in [D, §D.7.2].

[5] The nerve of this groupoid is the Čech nerve of the epimorphism $G^0 \twoheadrightarrow \mathfrak{G}$.

A.5.2. *Strategy for constructing* $\mathbb{G}^\bullet$. Suppose that G is lifted to a smooth affine group scheme $\tilde{G}$ over $\mathbb{Z}_p$. Suppose that we have a ring scheme $\mathcal{A}$ over Spf $\mathbb{Z}_p$ equipped with an epimorphism

$$\mathcal{A} \twoheadrightarrow {}^s\mathscr{R}_n. \tag{A.5}$$

Set $\mathbb{G}^0 := \tilde{G}(\mathcal{A})$. Then we get a homomorphism

$$\mathbb{G}^0 = \tilde{G}(\mathcal{A}) \to \tilde{G}({}^s\mathscr{R}_n) = G({}^s\mathscr{R}_n) = \mathfrak{G}. \tag{A.6}$$

Using smoothness of $\tilde{G}$, one checks that it is surjective (in the sense of fpqc sheaves). So one gets a crossed module $\mathbb{G}^\bullet$ by applying §A.5.1(i) to the homomorphism (A.6). One has $\mathbb{G}^{-1} = \operatorname{Ker}(\mathbb{G}^0 \twoheadrightarrow \mathfrak{G})$.

A.5.3. *A nice crossed module.* There exists an epimorphism (A.5) with $\mathcal{A} = W_n$, where W_n is the ring scheme[6] of n-truncated p-typical Witt vectors. Moreover, if $p > 2$ then there is a very nice epimorphism

$$W_n \twoheadrightarrow {}^s\mathscr{R}_n \tag{A.7}$$

whose kernel equals $\hat{W}^{(F^n)} := \operatorname{Ker}(F^n : \hat{W} \to \hat{W})$. Here $\hat{W} \subset W$ is the following ind-scheme: for any $A \in$ p-Nilp, the ideal $\hat{W}(A)$ is the set of all $x \in W(A)$ such that all components of the Witt vector x are nilpotent and all but finitely many of them are zero.

The crossed module $\mathbb{G}^\bullet$ corresponding to (A.7) is very simple: $\mathbb{G}^0 = \tilde{G}(W_n)$, $\mathbb{G}^{-1} = \tilde{G}(\hat{W}^{(F^n)})$, $d : \mathbb{G}^{-1} \to \mathbb{G}^0$ comes from the canonical map $\hat{W}^{(F^n)} \to W_n$, and the action of $\mathbb{G}^0$ on $\mathbb{G}^{-1}$ comes from the equality $\mathbb{G}^{-1} = \operatorname{Ker}(\tilde{G}(W_n \ltimes \hat{W}^{(F^n)}) \twoheadrightarrow \tilde{G}(W_n))$, where $W_n \ltimes \hat{W}^{(F^n)}$ is the semidirect product.

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[6] W_n is *not* a scheme of $\mathbb{Z}/p^n\mathbb{Z}$-algebras (this is why we need $\tilde{G}$). On the other hand, $W_{n,\mathbb{F}_p}$ *is* a scheme of $\mathbb{Z}/p^n\mathbb{Z}$-algebras.