SUPERSPECIAL REPRESENTATIONS OF WEYL GROUPS

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Dedicated to Gérard Laumon with admiration

Introduction

0.1. Let W be a Weyl group. Let $\text{Irr}(W)$ be the set of (isomorphism classes of) irreducible representations over $\mathbf{C}$ of W.

In this paper we define a subset $\text{Irr}_{ssp}(W)$ of the set of special representations of W. (See 1.3, 1.14.)

To do this we consider a connected reductive group G with Weyl group W defined and split over a finite field F_q with group of rational points $G(F_q)$ and also an F_q-rational structure on G for which the Frobenius acts on the Weyl group as opposition, with group of rational points $G(F_q)'$. From [L84] it is known that (for sufficiently large q) there is a bijection $\rho \mapsto \rho'$ from the set of unipotent representationsof $G(F_q)$ to the set of unipotent representations of $G(F_q)'$ such that ± 1 times the dimension of ρ' (as a polynomial in q) is obtained by replacing q by $-q$ in the polynomial in q which gives the dimension of ρ. Our observation is that there is at most one $\rho = \rho_E$ in the unipotent principal of $G(F_q)$ series which corresponds to a special representation E of W and is such that ρ' is cuspidal for $G'(F_q)$. The E obtained in this way are called superspecial. The irreducible Weyl groups for which such E exist are listed in 1.17. It turns out that if E is as above then $\dim(\rho_E)$ is of the form $q^{a_E}\sharp(G(F_q))^*/(\pm c_E P_E(-q))$ where a_E, c_E are independent of q, $\sharp(G(F_q))^*$ is the part prime to q in $\sharp(G(F_q))$ and P_E is a polynomial in q with coefficients in $\mathbf{N}$; moreover, P_E can be factored as a product of remarkably simple polynomials (each one with coefficients in $\mathbf{N}$).

Although the arguments above were our motivation, our definition of superspecial representations is actually purely in terms of generic degrees and does not make use of the groups $G(F_q)$ or $G'(F_q)$.

In §2 we associate to a superspecial representation of W (assumed to be irreducible) a constructible representation Z_W of W (or, in the nonsimply laced case, two such representations, Z_W, Z'_W.) We will show elsewhere that using these representations one can reconstruct (without using algebraic geometry) the finite

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groups (products of symmetric groups) associated in [L84] to any special representation, which were used in [L84] to classify unipotent representations of reductive groups over F_q.

In §3 the definition of superspecial representations is extended to finite noncrystallographic Coxeter groups.

In §4 we associate (using [L02]) to a superspecial representation of W (assumed to be irreducible with trivial opposition) an elliptic conjugacy class of W, which we call the superspecial conjugacy class.

0.2. Notation. For $n \in \mathbf{Z}$ we set $[n/2] = n/2$ if $n \in 2\mathbf{Z}$ and $[n/2] = (n-1)/2$ if $n \in 2\mathbf{Z}+1$.

1. Definition of superspecial representations

1.1. The set of simple reflections of W is denoted by I. Let r be the number of orbits of the opposition involution $op : I \to I$.

Let u be an indeterminate. For $E \in \text{Irr}(W)$ the generic degree $D_E(u)$ can be defined in terms of the Iwahori-Hecke algebra of W (see for example [AL,p.202]). (A priori, $D_E(u)$ is in the quotient field of $\mathbf{C}[u]$; in fact, it is in $\mathbf{Q}[u]$.) It is known that

(a) $$D_E(u) = u^{a_E} \prod_{i=1}^{r} (u^{e_i+1} - 1)/((-1)^{degP_E} c_E P_E(-u))$$

where $e_1, e_2, \dots, e_r$ are the exponents of W, $a_E \in \mathbf{N}$, $c_E \in \{1, 2, 3, \dots\}$ and $P_E(u)$ is a product of cyclotomic polynomials.

1.2. We say that E is a *special representation* if E appears in the a_E-th symmetric power of the reflection representation of W. (It then appears there with multiplicity 1.)

Let $\text{Irr}_{sp}(W)$ be the subset of $\text{Irr}(W)$ consisting of special representations. (This subset has been introduced in [L79a].)

In [L79], the set $\text{Irr}(W)$ has been partitioned into *families*; it is known that any family contains a unique special representation.

For $E \in \text{Irr}(W)$ we denote by γ_E the largest integer such that $(1+u)^{\gamma_E}$ divides the polynomial $D_E(u)$.

Theorem 1.3. *(i) For any $E \in \text{Irr}(W)$ we have $\gamma_E \le r$.*

(ii) The set $\text{Irr}_{ssp}(W) := \{E \in \text{Irr}_{sp}(W); \gamma_E = r\}$ consists of at most one element.

(iii) If $E \in \text{Irr}_{ssp}(W)$ then $P_E(u) \in \mathbf{N}[u]$. It has degree $2a_E + \sharp(I)$ and is a product of polynomials of the form $1+u^s+u^{2s}+\dots+u^{(l-1)s}$ where $s \in \{1, 2, 3, \dots\}$ and l is a prime number dividing $2c_E$.

To prove the theorem we can assume that W is irreducible. The various cases will be considered in 1.6-1.13.

We say that $E \in \mathrm{Irr}(W)$ is *superspecial* if it is contained in $\mathrm{Irr}_{ssp}(W)$. We say that W is *superspecial* if $\mathrm{Irr}_{ssp}(W) \neq \emptyset$; we then denote by E_W the unique object of $\mathrm{Irr}_{ssp}(W)$ and by $\mathcal{F}_W$ the family of $\mathrm{Irr}(W)$ that contains E_W.

1.4. Let $X \subset \mathbf{Z}_{>0}$, $\sharp(X) = m < \infty$. We set

$$n = \sum_{\lambda \in X} \lambda - \binom{m}{2}.$$

Note that $n > 0$. Let $X^0 = X \cap 2\mathbf{N}, X^1 = X \cap (2\mathbf{N}+1)$, $m^0 = \sharp(X^0), m^1 = \sharp(X^1)$. Let

(a) $$\alpha = \binom{m^0}{2} - \sum_{\lambda \in X^0} \lambda/2 + \binom{m^1}{2} - \sum_{\lambda \in X^1} \lambda/2 + m^1/2 + [n/2].$$

We show:

(b) We have $\alpha \le [n/2]$ with equality if and only if $X = \{1, 3, \dots, 2m-1\}$.

We have

$$\binom{m^0}{2} - \sum_{\lambda \in X^0} \lambda/2 \le (1+2+\dots+(m^0-1)) - (2+4+\dots+(2m^0))/2 \le -m^0 \le 0$$

with the last $\le$ being $=$ if and only if $m^0 = 0$. We have

$$\binom{m^1}{2} + m^1/2 - \sum_{\lambda \in X^1} \lambda/2 \le (1+3+\dots+(2m^1-1))/2 - (1+3+\dots+(2m^1-1))/2 = 0$$

with $\le$ being $=$ if and only if $X^1 = \{1, 3, \dots, 2m^1 - 1\}$. Since $m = m^0 + m^1$, this proves (b).

1.5. Let $(X, Y) \subset \mathbf{N} \times \mathbf{N}$ be such that $0 \notin Y$, $x = \sharp(X) < \infty$, $y = \sharp(Y) < \infty$. We set $m = x + y$,

$$n = \sum_{\lambda \in X} \lambda + \sum_{\lambda \in Y} \lambda - [(m-1)^2/2]/2.$$

Let

$$X^0 = X \cap 2\mathbf{N}, X^1 = X \cap (2\mathbf{N}+1), x^0 = \sharp(X^0), x^1 = \sharp(X^1),$$
$$Y^0 = Y \cap 2\mathbf{N}, Y^1 = Y \cap (2\mathbf{N}+1), y^0 = \sharp(Y^0), y^1 = \sharp(Y^1).$$

Let

(a) $$\alpha = \binom{x^0}{2} + \binom{x^1}{2} + \binom{y^0}{2} + \binom{y^1}{2} + x^0y^1 + x^1y^0 - \sum_{\lambda \in X} \lambda - \sum_{\lambda \in Y} \lambda + 2[n/2].$$

We show:

(b) We have $\alpha \le 2[n/2]$ with equality if and only if

$$(X,Y) = (\{0,2,4,\dots,m-2\},\{1,3,\dots,m-1\})$$

(with m even) or

$$(X,Y) = (\{0,2,4,\dots,m-1\},\{1,3,\dots,m-2\})$$

(with m odd).
We have $\alpha = \alpha^{01} + \alpha^{10} + 2[n/2]$ where

$$\alpha^{01} = \binom{x^0+y^1}{2} - \sum_{\lambda \in X^0 \sqcup Y^1} \lambda,$$

$$\alpha^{10} = \binom{x^1+y^0}{2} - \sum_{\lambda \in X^1 \sqcup Y^0} \lambda.$$

We have

$$\alpha^{01} \le z$$

where

$$z = (1+2+3+\dots+(x^0+y^1-1)) - (0+2+4+\dots+(2x^0-2)) - (1+3+\dots+(2y^1-1).$$

If $X^0 = \{0,2,4,\dots,2x^0-2\}, Y^1 = \{1,3,\dots,2y^1-1\}$, we have $\alpha^{01} = z$; if this condition is not satisfied then $\alpha^{01} < z$ Now $z = 0$ if $x^0 = y^1$ or $x^0 = y^1+1$ and $z < 0$ in all other cases. We see that $\alpha^{01} = 0$ if

$$X^0 = \{0,2,4,\dots,2x^0-2\}, Y^1 = \{1,3,\dots,2x^0-1\}$$

(for some $x_0 \ge 0$) or if

$$X^0 = \{0,2,4,\dots,2x^0-2\}, Y^1 = \{1,3,\dots,2x^0-3\}$$

(for some $x_0 \ge 1$) and $\alpha^{01} < 0$ in all other cases.
Similarly, $\alpha^{10} = 0$ if

$$Y^0 = \{0,2,4,\dots,2y^0-2\}, X^1 = \{1,3,\dots,2y^0-1\}$$

(and $y_0 \ge 0$ must be 0 since $0 \notin Y^0$) or if

$$Y^0 = \{0,2,4,\dots,2y^0-2\}, X^1 = \{1,3,\dots,2y^0-3\}$$

(for some $y_0 \ge 1$, but this would imply $0 \in Y^0$ which is not the case); we have $\alpha^{10} < 0$ in all other cases. In other words, we have $\alpha^{10} = 0$ if $Y^0 = \emptyset, X^1 = \emptyset$ (that is $y^0 = x^1 = 0$) and $\alpha^{10} < 0$ in all other cases. When $x^0 = y^1$, $y^0 = x^1 = 0$ we have $x^0 = y^1 = m/2$ so that $m \in 2\mathbf{N}$. When $x^0 = y^1+1$, $y^0 = x^1 = 0$ we have $x^0 = (m+1)/2, y_0 = (m-1)/2$ so that $m \in 2\mathbf{N}+1$. This proves (b).

1.6. In this subsection we assume that W is of type A_{n-1} for some $n \geq 1$. Then $\mathrm{Irr}(W)$ is indexed as in [L84] by the various $X \subset \mathbf{Z}_{>0}$ as in the beginning of 1.4. By the formulas for $D_E(u)$ in [L84,p.358] we see that if $E \in \mathrm{Irr}(W)$ corresponds to X then γ_E is equal to α in 1.4(a). Then 1.3(i),(ii) follow from 1.4(b).

We see that the condition that $\mathrm{Irr}_{ssp}(W) \neq \emptyset$ is that $n = (k^2+k)/2$ for some $k \geq 1$; for such n the unique element $E \in \mathrm{Irr}_{ssp}(W)$ corresponds to $X = \{1, 3, \dots, 2k-1\}$; we have $c_E = 1$,

$$P_E(u) = (1+u)^k (1+u^3)^{k-1} (1+u^5)^{k-2} \dots (1+u^{2k-1})/(1+u).$$

We see that 1.3(iii) holds.

1.7. In this subsection we assume that W is of type B_n for some $n \geq 2$. Then $\mathrm{Irr}(W)$ is indexed as in [L84] by the various (X, Y) as in the beginning of 1.5 such that $\sharp(X) = \sharp(Y) + 1$. By the formulas for $D_E(u)$ in [L84,p.359] we see that if $E \in \mathrm{Irr}(W)$ corresponds to (X, Y) then γ_E is equal to α in 1.5(a). Then 1.3(i),(ii) follow from 1.5(b).

We see that the condition that $\mathrm{Irr}_{ssp}(W) \neq \emptyset$ is that $n = k^2 + k$ for some k; for such n the unique element $E \in \mathrm{Irr}_{ssp}(W)$ corresponds to

$$(X, Y) = (\{0, 2, 4, \dots, 2k\}, \{1, 3, \dots, 2k-1\}).$$

We have $c_E = 2^k$,

$$P_E(u) = (1+u)^{2k} (1+u^2)^{2k-1} \dots (1+u^{2k-1})^2 (1+u^{2k}).$$

We see that 1.3(iii) holds. We have $deg(P_E) = 2k(k+1)(2k+1)/3$.

1.8. In this subsection we assume that W is of type D_n for some $n \geq 4$. Then $\mathrm{Irr}(W)$ is indexed as in [L84] by the various (X, Y) as in the beginning of 1.5 such that $\sharp(X) = \sharp(Y)$ except that there are two representations corresponding to any pair of the form (X, Y) with $X = Y$.

By the formulas for $D_E(u)$ in [L84,p.359] we see that if $E \in \mathrm{Irr}(W)$ corresponds to (X, Y) then γ_E is equal to α in 1.5(a). Then 1.3(i),(ii) follow from 1.5(b).

We see that the condition that $\mathrm{Irr}_{ssp}(W) \neq \emptyset$ is that $n = k^2$ for some k; for such n the unique element $E \in \mathrm{Irr}_{ssp}(W)$ corresponds to

$$(X, Y) = (\{0, 2, 4, \dots, 2k-2\}, \{1, 3, \dots, 2k-1\}).$$

We have $c_E = 2^{k-1}$,

$$P_E(u) = (1+u)^{2k-1} (1+u^2)^{2k-2} \dots (1+u^{2k-2})^2 (1+u^{2k-1}).$$

We see that 1.5(iii) holds. We have $deg(P_E) = k(4k^2-1)/3$.

1.9. In this subsection we assume that W is of type E_6. Using the table in [L84,p.363], we see that 1.3(i),(ii) hold; $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E)=80$. We have $c_E=6$, $a_E=7$,

$$P_E(u)=(1+u)^3(1+u^2)^2(1+u^3)^3(1+u+u^2)^2.$$

Hence 1.3(iii) holds. We have $deg(P_E)=20$.

1.10. In this subsection we assume that W is of type E_7. Using the table in [L84,p.364,365] we see that 1.3(i),(ii) hold; $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E)=512$, $a_E=11$. We have $c_E=2$,

$$P_E(u)=(1+u)^2(1+u^3)^2(1+u^5)(1+u^7)(1+u^9).$$

Hence 1.3(iii) holds. We have $deg(P_E)=29$.

1.11. In this subsection we assume that W is of type E_8. Using the table in [L84,p.366-369] we see that 1.3(i),(ii) hold; $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E)=4480$. We have $c_E=120$, $a_E=16$,

$$P_E(u)=(1+u)^4(1+u^2)^4(1+u^3)^4(1+u+u^2)^4(1+u+u^2+u^3+u^4)^2.$$

Hence 1.3(iii) holds. We have $deg(P_E)=40$.

1.12. In this subsection we assume that W is of type F_4. Using the table in [L84,p.371] we see that 1.3(i),(ii) hold; $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E)=12$. We have $c_E=24$, $a_E=4$,

$$P_E(u)=(1+u)^4(1+u^2)^2(1+u+u^2)^2.$$

Hence 1.3(iii) holds. We have $deg(P_E)=12$.

1.13. In this subsection we assume that W is of type G_2. Using the table in [L84,p.372] we see that 1.3(i),(ii) hold; $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E)=2$, $c_E=6$. We have $a_E=1$,

$$P_E(u)=(1+u)^2(1+u+u^2).$$

Hence 1.3(iii) holds. We have $deg(P_E)=4$.

This completes the proof of Theorem 1.3.

1.14. In the case where W is irreducible of type $\neq A, E_7$, the condition that W is superspecial is equivalent to the following condition:

(a) There exists $E\in\mathrm{Irr}_{sp}(W)$ such that for any $E'\in\mathrm{Irr}_{sp}(W)-\{E\}$ we have $c_{E'}<c_E$.

(Then E is unique and is equal to E_W.)

1.15. It may happen that some non-special $E\in\mathrm{Irr}(W)$ satisfy $\gamma_E=r$. For W of type E_8, the representations $7168_w, 2688_y$ (notation of [L84]) are such examples; for W of type F_4, the representations $4_1, 16_1$ (notation of [L84]) are such examples. But for other irreducible W there are no such examples.

1.16. We can write $W=W_1\times W_2\times\ldots\times W_e$ where $W_1, W_2, \ldots, W_e$ are irreducible Weyl groups. If E_i, E'_i are in $\mathrm{Irr}(W_i)$ $(i=1,\ldots,e)$ then $E_1\boxtimes E_2\boxtimes\ldots\boxtimes E_e$, $E'_1\boxtimes E'_2\boxtimes\ldots\boxtimes E'_e$ are in the same family of $\mathrm{Irr}(W)$ if and only if E_i, E'_i are in the same family of $\mathrm{Irr}(W_i)$ for $i=1,\ldots,e$; we have $E_1\boxtimes E_2\boxtimes\ldots\boxtimes E_e\in\mathrm{Irr}_{sp}(W)$ if and only if $E_i\in\mathrm{Irr}_{sp}(W_i)$ for $i=1,\ldots,e$; we have $E_1\boxtimes E_2\boxtimes\ldots\boxtimes E_e\in\mathrm{Irr}_{ssp}(W)$ if and only if $E_i\in\mathrm{Irr}_{ssp}(W_i)$ for $i=1,\ldots,e$.

1.17. Here is the list of superspecial Weyl groups W that are irreducible or $\{1\}$.
$A_{(k^2+k)/2-1}$ $(k\in\{1,2,3,\ldots\})$;
B_{k^2+k} $(k\in\{1,2,3,\ldots\})$;
D_{k^2}, $k\in\{2,3,4,\ldots\}$;
E_6, E_7, E_8, F_4, G_2.
We note that in each case (except in type A and E_7) we have that r is even.

2. The representations Z_W, Z'_W

2.1. For any $I'\subset I$ we denote by $W_{I'}$ the subgroup of W generated by I'; this is again a Weyl group. For $E'\in\mathrm{Irr}(W_I)$ we recall that the truncated induction $J_{W_I}^W(E')$ is the representation of W in which the the multiplicity of any $E\in\mathrm{Irr}(W)$ is equal to the multiplicity of E in the ordinary induction $\mathrm{ind}_{W_I}^W(E')$ if $a_E=a_{E'}$ and is 0 if $a_E\ne a_{E'}$.

2.2. In this subsection we assume that W is irreducible or $\{1\}$, superspecial, simply laced. We will associate to W a representation Z_W of W of the form $Z_W=E_1\oplus E_2\oplus\ldots\oplus E_t$ where $E_1, E_2, \ldots, E_t$ are distinct irreducible representations in $\mathcal{F}_W$ satisfying the identity

(a) $$c_{E_1}^{-1}+c_{E_2}^{-1}+\cdots+c_{E_t}^{-1}=1$$

and (in the case where W is of type $\ne A$) the identity

(a1) $$(-1)^{b_{E_1}}c_{E_1}^{-1}+(-1)^{b_{E_2}}c_{E_2}^{-1}+\cdots+(-1)^{b_{E_t}}c_{E_t}^{-1}=0$$

where $b_E\in\mathbf{N}$ is defined as in [L84,(4.1.2)].

If W is of type A (this includes the case $W=\{1\}$) there is a unique choice for such Z_W namely $Z_W=E_W$ (recall that $c_{E_W}=1$).

If W is of type E_7 there is again a unique choice for such Z_W namely $Z_W=E_1\oplus E_2$ where E_1, E_2 are the two objects of $\mathcal{F}_W$. The identities (a),(a1) are now $2^{-1}+2^{-1}=1$. $2^{-1}-2^{-1}=0$.

We now assume that W is of type $\ne A, E_7$.

(b) There is a unique choice of $i \in I$ such that $I - \{i\} = I' \sqcup I''$, $W_{I-\{i\}} = W_{I'} \times W_{I''}$ with $W_{I'}$ superspecial, irreducible or $\{1\}$ and with $W_{I''}$ a product of Weyl groups of type A and such that E_W appears with nonzero multiplicity in

$$J_{W_{I-\{i\}}}^W(E_{W_{I'}} \boxtimes \mathrm{sgn}_{W_{I''}}).$$

We define

(c) $$Z_W = J_{W_{I-\{i\}}}^W(Z_{W_{I'}} \boxtimes \mathrm{sgn}_{W_{I''}}).$$

(We can assume that $Z_{W_{I'}}$ is known by induction.) We now describe Z_W in the various cases.

If W is of type D_{k^2} (with $k \geq 2$) we have that $W_{I'}$ is of type $D_{(k-1)^2}$ (if $k \geq 3$), $I' = \emptyset$ (if $k = 2$) and $W_{I''}$ is of type A_{2k-2} (if $k \geq 3$) and of type $A_1 \times A_1 \times A_1$ (if $k = 2$).

We have $Z_W = \oplus_\sigma E_\sigma$ where σ runs over all permutations of $1, 2, 3, \ldots, 2k-2$ which preserve each of the unordered pairs $(1,2), (3,4), \ldots, (2k-3, 2k-2)$ and for such σ, $E_\sigma \in \mathrm{Irr}(W)$ corresponds as in 1.8 to

$$(X, Y) = (\{0, \sigma(2), \sigma(4), \ldots, \sigma(2k-2)\}, \{\sigma(1), \sigma(3), \ldots, \sigma(2k-3), 2k-1\}).$$

Note that $c_{E_\sigma} = 2^{k-1}$ for any σ hence $\sum_\sigma c_{E_\sigma}^{-1} = 1$. The identity (a) is now $2^{-k+1} + 2^{-k+1} + \cdots + 2^{-k+1} = 1$ (the sum has 2^{k-1} terms.)

For $E_\sigma, (X, Y)$ as above we have $(-1)^{b_{E_\sigma}} = (-1)^{\sum_{j \in Y} j} h$ where $h = \pm 1$ is independent of σ (see [L79a]); hence the identity (a1) holds.

If W is of type E_6 we have $I' = \emptyset$ and $W_{I''}$ of type $A_2 \times A_2 \times A_1$. We have $Z_W = 80_7 + 60_8 + 10_9$ (notation of [L15,4.4]. We have $c_{80_7} = 6, c_{60_8} = 2, c_{10_9} = 3$. The identities (a),(a1) are now $6^{-1} + 2^{-1} + 3^{-1} = 1$, $6^{-1} - 2^{-1} + 3^{-1} = 0$.

If W is of type E_8 we have $I' = \emptyset$ and $W_{I''}$ of type $A_4 \times A_3$. We have

$$Z_W = 4480_{16} + 3150_{18} + 4200_{18} + 420_{20} + 7168_{17} + 1344_{19} + 2016_{19}$$

(notation of [L15,4.4]). We have
$c_{4480_{16}} = 120, c_{3150_{18}} = 6, c_{4200_{18}} = 8, c_{420_{20}} = 5,$
$c_{7168_{17}} = 12, c_{1344_{19}} = 4, c_{2016_{19}} = 6.$
The identities (a),(a1) are now
$120^{-1} + 6^{-1} + 8^{-1} + 5^{-1} + 12^{-1} + 4^{-1} + 6^{-1} = 1,$
$120^{-1} + 6^{-1} + 8^{-1} + 5^{-1} - 12^{-1} - 4^{-1} - 6^{-1} = 0.$

In each case, Z_W is a constructible representation of W (or a cell in the sense of [L82]); moreover each irreducible component of Z_W is 2-special (in the sense if [L15]).

2.3. In this subsection we assume that W is irreducible, superspecial, not simply laced. We will associate to W two representations Z_W, Z'_W of W of the form

$Z_W = E_1 \oplus E_2 \oplus \dots \oplus E_t$, $Z'_W = E'_1 \oplus E'_2 \oplus \dots \oplus E'_t$

where $E_1, E_2, \dots, E_t$ (resp. $E'_1, E'_2, \dots, E'_t$) are distinct irreducible representations in $\mathcal{F}_W$ satisfying

(a) $$c_{E_1}^{-1} + c_{E_2}^{-1} + \dots + c_{E_t}^{-1} = 1$$

(a1) $$(-1)^{b_{E_1}} c_{E_1}^{-1} + (-1)^{b_{E_2}} c_{E_2}^{-1} + \dots + (-1)^{b_{E_t}} c_{E_t}^{-1} = 1$$

and $c_{E_1} = c_{E'_1}, \dots, c_{E_t} = c_{E'_t}$, $b_{E_1} = b_{E'_1}, \dots, b_{E_t} = b_{E'_t}$.

Now statement 2.2(b) remains true in our case except when W is of type B_2, G_2 or F_4 in which case 2.2(b) is true if one replaces "unique choice of $i \in I$" by "exactly two choices of $i \in I$".

If W is of type B_2, G_2 or F_4 we define Z_W as in 2.2(c) using one of the two choices of i as above; in these cases we have $I' = \emptyset$. The same definition using the second choice of i gives a second representation denoted by Z'_W. If W is of type B_2, we have $Z_W = 2_1 + 1_2$, $Z'_W = 2_1 + 1_2$ (notation of [L15, 4.4]; the 1_2 in Z_W is different from the 1_2 in Z'_W. Then (a),(a1) become $2^{-1} + 2^{-1} = 1$, $2^{-1} - 2^{-1} = 0$.

If W is of type G_2, we have $Z_W = 2_1 + 2_2 + 1_3$ $Z'_W = 2_1 + 2_2 + 1_3$ (notation of [L15,4.4]); the 1_3 in Z_W is different from the 1_3 in Z'_W. Then (a),(a1) become $6^{-1} + 2^{-1} + 3^{-1} = 1$, $6^{-1} - 2^{-1} + 3^{-1} = 0$.

If W is of type F_4, we have

$Z_W = 12_4 \oplus 6_6 \oplus 9_6 \oplus 4_7 \oplus 16_5$,

$Z'_W = 12_4 \oplus 6_6 \oplus 9_6 \oplus 4_7 \oplus 16_5$, (notation of [L15,4.4]; the $9_6, 4_7$ in Z_W are different from the $9_6, 4_7$ in Z'_W. Then (a),(a1) become $24^{-1} + 3^{-1} + 8^{-1} + 4^{-1} + 4^{-1} = 1$, $24^{-1} + 3^{-1} + 8^{-1} - 4^{-1} - 4^{-1} = 0$.

When W is of type $B_{k^2+k}, k \geq 2$, statement 2.2(b) remains true. We define

$$Z_W = J_{W_{I-\{i\}}}^W (Z_{W_{I'}} \boxtimes \mathrm{sgn}_{W_{I''}}),$$

$$Z'_W = J_{W_{I-\{i\}}}^W (Z'_{W_{I'}} \boxtimes \mathrm{sgn}_{W_{I''}}),$$

(We can assume that $Z_{W_{I'}}, Z'_{W_{I'}}$ are known by induction.)

We have $Z_W = \oplus_\sigma E_\sigma$ where σ runs over all permutations of $1, 2, 3, \dots, 2k$ which preserve each of the unordered pairs $(1,2), (3,4), \dots, (2k-1, 2k)$ and for such σ, $E_\sigma \in \mathrm{Irr}(W)$ corresponds as in 1.7 to

$$(X, Y) = (\{0, \sigma(2), \sigma(4), \dots, \sigma(2k)\}, \{\sigma(1), \sigma(3), \dots, \sigma(2k-1)\}).$$

Note that $c_{E_\sigma} = 2^k$ for any σ hence $\sum_\sigma c_{E_\sigma}^{-1} = 1$. For $E_\sigma, (X, Y)$ as above we have $(-1)^{b_{E_\sigma}} = (-1)^{\sum_{j \in Y} j} h$ where $h = \pm 1$ is independent of σ (see [L79a]); hence the identity (a1) holds.

We have $Z'_W = \oplus_\sigma E'_\sigma$ where σ runs over all permutations of $0, 1, 2, 3, \dots, 2k-1$ which preserve each of the unordered pairs $(0, 1), (2, 3), \dots, (2k-2, 2k-1)$ and for such σ, $E'_\sigma \in \mathrm{Irr}(W)$ corresponds as in 1.7 to

$$(X, Y) = (\{\sigma(0), \sigma(2), \sigma(4), \dots, \sigma(2k-2), 2k\}, \{\sigma(1), \sigma(3), \dots, \sigma(2k-1)\}).$$

In each case, Z_W, Z'_W are constructible representations of W (or cells in the sense of [L82]) such that $Z'_W = Z_W \otimes \mathrm{sgn}_W$; their irreducible components are 2-special (in the sense if [L15]).

3. The noncrystallographic case

3.1. In this section we assume that W is an irreducible noncrystallographic finite Coxeter group with set I of simple reflections. Now $\mathrm{Irr}(W)$ is defined as in 0.1; the generic degree $D_E(u)$ for $E \in \mathrm{Irr}(E)$ is defined as in 1.1. (We now have $D_E(u) \in \mathbf{R}[u]$, see [AL,p.202], [L82].) From the formulas for $D_E(u)$ in type H_4 in [AL] one can verify that an equality like 1.1(a) still holds except that now c_E is only an algebraic integer in $\mathbf{R}_{>0}$ and $P_E(u)$ is now only in $\mathbf{R}[u]$ with leading coefficient 1; a similar property holds in the cases $\ne H_4$. Then $a_E \in \mathbf{N}$ is defined as in 1.1. The definition of families in [L79] and that of $\mathrm{Irr}_{sp}(W)$ extend in an obvious way to our case. (The representations in $\mathrm{Irr}_{sp}(W)$ are described explicitly in type H_4 in [AL,§5].) For $E \in \mathrm{Irr}(W)$ we define $\gamma_E \in \mathbf{N}$ as in 1.2.

Theorem 3.2. *(i) For any $E \in \mathrm{Irr}(W)$ we have $\gamma_E \le \sharp(I)$.*

(ii) The set $\mathrm{Irr}_{ssp}(W) := \{E \in \mathrm{Irr}_{sp}(W); \gamma_E = \sharp(I)\}$ consists of exactly one element.

(iii) If $E \in \mathrm{Irr}_{ssp}(W)$ then $P_E(u) \in \mathbf{R}_{\ge 0}[u]$. It has degree $2a_E + \sharp(I)$.

This can be verified using the known results on $D_E(u)$. Assume first that W is of type H_4. Now $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E) = 24$, $a_E = 6$. We have $c_E = 120/(13 - 8\lambda) = 120(5 + 8\lambda)$,

$$P_E(u) = (u+1)^4(u^2+u+1)^2(u^2+\lambda u+1)^2(u^2-\tilde{\lambda}u+1)^2$$

where

$$\lambda = (1+\sqrt{5})/2 \in \mathbf{R}_{>0}, -\tilde{\lambda} = (\sqrt{5}-1)/2 \in \mathbf{R}_{>0}.$$

Assume next that W is of type H_3. Now $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E) = 4$, $a_E = 3$. We have $c_E = 2$, $P_E(u) = (1+u)(1+u^3)(1+u^5)$.

If W is a dihedral group of order $2p$, $p = 5$ or $p \ge 7$ then $\mathrm{Irr}_{ssp}(W)$ consists of the unique E such that $\dim(E) = 2$, $a_E = 1$. We have

$$c_E = p/((1-\xi)(1-\xi^{-1})) = \prod_{t=2}^{p-2}(1-\xi^t),$$

$P_E = (1+u)^2(1+(\xi+\xi^{-1})u+u^2)$ where $\xi = e^{2\pi\sqrt{-1}/p}$.

4. Superspecial conjugacy classes in W

4.1. In this section we assume that W is an irreducible superspecial Weyl group. Let $E = E_W$.

Until the end of 4.3 we assume also that the opposition $op : I \to I$ is the identity map.

Let $G, G(F_q), \rho_E, \rho'$ be as in 0.1. Recall that ρ' is a unipotent cuspidal representation of $G(F_q)$, say over $\bar{\mathbf{Q}}_l$. For any $w \in W$ let X_w be the subvariety of the flag manifold of G defined in [DL] (it consists of Borel subgroups which are in relative position w with their transform under the Frobenius map. Let $H^i_c(X_w)$ be the i-th l-adic cohomology with compact support of X_w viewed as a representation of $G(F_q)$. Let $c(W)$ be the set of all $w \in W$ such that ρ' appears with nonzero multiplicity in the virtual representation $\sum_i H^i_c(X_w)$ of $G(F_q)$. Let $M(W)$ be the minimum of the legths of various elements in $c(W)$ and let $c_{min}(W)$ be the set of elements of $c(W)$ of length $M(W)$. The following result can be deduced from [L02,2.18].

Theorem 4.2. *There is a unique conjugacy class C_W in W such that $c_{min}(W) \subset C_W$.*

The conjugacy class C_W is said be the superspecial conjugacy class of W. It is an elliptic conjugacy class.

4.3. We describe C_W in each case (we use [L02]). We also describe in each case the number $M(W)$ (we use [GP]).

If W is of type $B_{k^2+k}, k \ge 1$ viewed in an obvious way as a subgroup of $S_{2(k^2+k)}$ then C_W is the elliptic conjugacy class with cycle type $4 + 8 + 12 + \cdots + 4k$. We have $M(W) = k(k + 1)(2k + 1)/3$.

If W is of type D_{k^2}, $k \ge 2$ even, viewed in an obvious way as a subgroup of S_{2k^2} then C_W is the elliptic conjugacy class with cycle type $2 + 6 + 10 + \cdots + (4k - 2)$. We have $M(W) = 2k(k^2 - 1)/3$.

If W is of type E_8 then C_W consists of elements with characteristic polynomial $(u^2 - u + 1)^4$ in the reflection representation. We have $M(W) = deg(P_E(u)) = 40$. We have $c(W) = c_{min}(W) = C_W$, $\sharp(C_W) = \dim(E)$.

If W is of type E_7 then C_W is the Coxeter conjugacy class. We have $M(W) = 7$.

If W is of type F_4 then C_W consists of elements with characteristic polynomial $(u^2 + 1)^2$ in the reflection representation. We have $M(W) = deg(P_E(u)) = 12$. We have $c(W) = c_{min}(W) = C_W$, $\sharp(C_W) = \dim(E)$.

If W is of type G_2 then C_W consists of elements with characteristic polynomial $u^2 + u + 1$ in the reflection representation. We have $M(W) = deg(P_E(u)) = 4$. We have $c(W) = c_{min}(W) = C_W$, $\sharp(C_W) = \dim(E)$.

4.4. We now assume that the opposition $op : I \to I$ is not the identity map. It induces an involution $op : W \to W$. Then results similar to 4.2 hold (we use [L02,2.19]). They associate to W a twisted conjugacy class in W (an orbit of the W-action $x : w \mapsto xwop(x)^{-1}$ on W).

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