

ALGEBRAIC FLAT CONNECTIONS AND O-MINIMALITY

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ABSTRACT. We prove that an algebraic flat connection has $\mathbb{R}_{\text{an},\text{exp}}$ -definable flat sections if and only if it is regular singular with unitary monodromy eigenvalues at infinity, refining previous work of Bakker–Mullane. This provides an o-minimal characterization of classical properties of the Gauss–Manin connection.

1. INTRODUCTION

Let X be a smooth complex algebraic variety. Let $\mathcal{E}$ be an algebraic vector bundle on X and $\nabla: \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ be a flat algebraic connection. We denote the analytic spaces underlying X and $\mathcal{E}$ by the same symbol and we endow them with their canonical $\mathbb{R}_{\text{an},\text{exp}}$ -definable structure, see [BBT23, Sec. 2 and 3]. Roughly speaking, a function on X is definable in this structure if it can be expressed in terms of the usual logical operators (including quantifiers), algebraic functions, real analytic functions on compact domains like $[0, 1]^n$ and the real exponential function. A fundamental property of this structure is its o-minimality, which essentially means that definable functions don't exhibit oscillatory behavior.

We say that $(\mathcal{E}, \nabla)$ has *definable flat sections* if any flat section $\sigma: U \rightarrow \mathcal{E}$ is definable for any definable open $U \subset X$.

Theorem 1.1. *The algebraic flat bundle $(\mathcal{E}, \nabla)$ has definable flat sections if and only if it is regular singular with unitary monodromy eigenvalues at infinity.*

Remarks 1.2. (1) The ‘if’ part is due to Bakker–Mullane [BM23, Thm. 1.2] and the ‘only if’ part refines their Example 3.3.

(2) One can easily show that the property of definable flat sections holds for all definable open subsets if and only if it holds for a single finite covering by simply connected open definable subsets (see the proof of Lemma 4.1). In addition, all flat sections on a given definable U are definable if and only if this property is true on a basis of such.

(3) The property of unitary monodromy eigenvalues at infinity can be defined in two equivalent ways:

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- (i) By considering a normal crossings compactification $\overline{X}$ of X and requiring that the eigenvalues of the monodromy action of a small loop around any irreducible component of $\overline{X} \setminus X$ have absolute value one (see [BM23, Def. 1.1]).
 - (ii) By requiring that for all non-singular complex curves C and all morphisms $\psi: C \rightarrow X$ the flat connection $\psi^*(\mathcal{E}, \nabla)$ has monodromy eigenvalues of absolute value one at each point of the smooth compactification $\overline{C}$ of C .
- (4) For the definition of regular singular algebraic connections see [Del70, Def. 4.5].

The ‘only if’ part of Theorem 1.1 is shown by reducing to a local problem in one variable in Section 4 and using the solution theory of irregular singular complex differential equations in Section 5. More precisely, we use the Multisummation Theorem, recalled in Section 2, and an o-minimal expansion of $\mathbb{R}_{\text{an}, \text{exp}}$ by ‘multisums’ due to van den Dries and Speissegger [vDS00], recalled in Section 3.

Our motivation for working out a proof of Theorem 1.1 comes from the study of the Gauss-Manin connection. In fact, Theorem 1.1 allows us to reformulate Griffiths’ theorem that the Gauss-Manin connection is regular singular [Gri70, Thm. (4.3)], together with the Griffiths–Landman–Grothendieck Monodromy Theorem [Gri70, Thm. (3.1)]. Those two theorems can now be expressed equivalently in terms of [BM23, Cor. 1.3]:

- ($\star$) The algebraic de Rham cohomology together with the Gauss-Manin connection $(\mathcal{H}_{\text{dR}}^j(Y/X), \nabla)$ of a smooth projective family $Y \rightarrow X$ of complex algebraic varieties has definable flat sections.

Observe that the Gauss-Manin local system is defined over $\mathbb{Z}$, so the eigenvalues of the local monodromies are of absolute value one if and only if they are roots of unity by Kronecker’s theorem.

In non-abelian Hodge theory, Simpson has suggested variants of these classical results from Hodge theory in terms of a logarithmic extension of the moduli space of vector bundles with relative flat connections $M_{\text{dR}}(Y/X) \rightarrow X$, see [Sim97, Sec. 8]. In a forthcoming paper we plan to address the

Question 1.3. Which flat sections of $M_{\text{dR}}(Y/X) \rightarrow X$ over open definable subsets of X are definable?

In the non-linear setting of non-abelian Hodge theory it is not clear to us whether all flat sections are definable or whether there is any direct connection to Simpson’s regularity result for the non-abelian Gauss-Manin connection [Sim97, Sec. 8] in the spirit of Theorem 1.1.

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2. REMINDER ON LINEAR MEROMORPHIC DIFFERENTIAL EQUATIONS

Let $\rho > 0$. We consider the meromorphic differential equation

$$(2.1) \quad zY' = A(z)Y$$

on the open disc $\Delta(\rho) = \{z \in \mathbb{C}, |z| < \rho\}$. Here A is an $r \times r$ -matrix with entries holomorphic functions on the punctured disc $\Delta(\rho)^\times = \Delta(\rho) \setminus \{0\}$ which are meromorphic along 0. In order to exclude undesirable behavior at the boundary of $\Delta(\rho)$, we assume that A is overconvergent, i.e. that all its entries extend to holomorphic functions on $\Delta(\tilde{\rho})^\times$ for some $\tilde{\rho} > \rho$.

Recall that a gauge transformation by P changes A to the coefficient matrix $P^{-1}AP - zP^{-1}P'$ in (2.1). A pullback along the finite covering $z \mapsto z^m$ for $m \in \mathbb{Z}_{\geq 1}$ replaces $A(z)$ by $mA(z^m)$. We say that A is in *split normal form* if there exist $Q_1(z), \dots, Q_\ell(z) \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$ and constant square matrices $B_1, \dots, B_\ell$ of size $r_1, \dots, r_\ell$ with $\sum_{i=1}^\ell r_i = r$ such that A is block diagonal with blocks $Q_1 \text{Id}_{r_1} + B_1, \dots, Q_\ell \text{Id}_{r_\ell} + B_\ell$. Note that for such a split normal form A a fundamental matrix solution on a sector has the shape

$$(2.2) \quad \mathcal{Y}(z) = \text{Diag}(z^{B_1} e^{\int \frac{Q_1(z)}{z} dz}, \dots, z^{B_\ell} e^{\int \frac{Q_\ell(z)}{z} dz}).$$

Theorem 2.1 (Fabry-Hukuhara-Turrittin-Levelt). *After pullback along a finite covering there exists a formal gauge transformation $P \in \text{GL}_r(\mathbb{C}((z)))$ which transforms the differential equation (2.1) into a split normal form.*

By first performing a meromorphic gauge transformation on A , we can assume without loss of generality that the formal gauge transformation P in Theorem 2.1 is in $\text{GL}_r(\mathbb{C}[[z]])$ and that $P(0)$ is the identity matrix. Note that the matrix P is itself a solution of a meromorphic differential equation

$$(2.3) \quad zP' = A(z)P - P\tilde{A}(z)$$

where $\tilde{A}$ is in split normal form.

Let $d \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be a direction and

$$D_d(\rho) = \{xd \mid x \in (0, \rho)\} \subset \Delta(\rho)$$

be its associated ray. By a *solution of (2.1) on the ray $D_d(\rho)$* we mean a function $Y: D_d(\rho) \rightarrow \mathbb{C}^r$ which extends to a holomorphic function on a sector in $\Delta(\rho)$ containing $D_d(\rho)$ which is a solution of (2.1) on that sector.

Definition 2.2. We call a function $f: D_d(\rho) \rightarrow \mathbb{C}$ a ‘*multisum*’ if

- f is the restriction to $D_d(\rho)$ of a real analytic function defined on $D_d(\rho')$ for some $\rho' > \rho$;
- for some $0 < \tilde{\rho} \leq \rho$ the function $f|_{D_d(\tilde{\rho})}$ is of the form

$$f = f_1|_{D_d(\tilde{\rho})} + \dots + f_n|_{D_d(\tilde{\rho})},$$

where f_j is a holomorphic function on a sector $\{z \in \Delta(\bar{\rho})^\times \mid |\arg(z/d)| < \kappa_j \phi_j\}$ with $\kappa_j \in (0, 1)$, $\phi_j \in (\frac{\pi}{2}, \pi)$ for which there exists $C > 0$ such that

$$\left| \frac{1}{n!} f_j^{(n)}(z) \right| \leq C^n (n!)^{\kappa_j}$$

for all $n \geq 0$ and all z in the above sector;

- the Taylor coefficients $\lim_{z \rightarrow 0} f_j^{(m)}(z)$ exist for the above f_j for all $m \geq 0$.

The set of all ‘multisums’ is a $\mathbb{C}$ -algebra closed under $\frac{d}{dz}$ and a ‘multisum’ is uniquely characterized by its asymptotic Taylor coefficients at $z = 0$. An important point is that the Taylor series at $z = 0$ might be not convergent.

A formal power series in $\mathbb{C}[[z]]$ is *multisummable* in direction d if it is the Taylor series at $z = 0$ of a ‘multisum’ on $D_d(\rho)$ for some $\rho > 0$. A basic reference for multisummable power series is [Bal94, Ch. 10].

Recall that the *slopes* of the differential equation (2.1) may be defined as

$$\left\{ \frac{1}{m} \deg_z Q_i \left(\frac{1}{z} \right), i = 1, \dots, \ell \right\}$$

where m is the degree of the necessary covering to achieve the split normal form in Theorem 2.1. See [vPS03, Ch. 3].

Theorem 2.3 (Multisummation Theorem). *Let $Y \in (\mathbb{C}[[z]])^r$ be a formal solution of a differential equation (2.1) all of whose positive slopes are $> \frac{1}{2}$. Then for all but a finite number of directions d , the components of Y are multisummable in the direction d .*

For a proof of the Multisummation Theorem see for instance [vPS03, Ch. 7].

Remarks 2.4.

- The condition on the slopes in Theorem 2.3 is satisfied after pullback along a finite covering $z \mapsto z^m$ for large m .
- Note that the uniqueness of the ‘multisum’ with given asymptotic Taylor expansion implies that the associated ‘multisums’ in Theorem 2.3 automatically satisfy the differential equation (2.1) and can be defined on $D_d(\rho)$ for the given ρ in (2.1).

Combining Theorem 2.1 for the differential equation (2.1) and Theorem 2.3 for the differential equation (2.3) with the above remarks we deduce:

Corollary 2.5. *After taking the pullback along a finite covering $z \mapsto z^m$ for some m and performing a meromorphic gauge transformation, there exist $Q_1(z), \dots, Q_L(z) \in \frac{1}{z} \mathbb{C}[\frac{1}{z}]$ and constant Jordan blocks $B_1, \dots, B_L$ with eigenvalues $b_1, \dots, b_L$ such that, except for a finite number of directions d , there exists $P_d: D_d(\rho) \rightarrow \mathrm{GL}_r(\mathbb{C})$ with entries being ‘multisums’, with $P_d(0)$ the identity matrix, such that*

$$\mathcal{Y}: D_d(\rho) \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad z \mapsto P_d(z) \mathrm{Diag} \left(z^{B_1} e^{\int \frac{Q_1(z)}{z} dz}, \dots, z^{B_L} e^{\int \frac{Q_L(z)}{z} dz} \right)$$

is a fundamental matrix solution of the differential equation (2.1) on $D_d(\rho)$.

The differential equation is regular singular if and only if $Q_1(z), \dots, Q_L(z)$ all vanish. In that case

$$\exp(2\pi i b_1), \dots, \exp(2\pi i b_L)$$

are the eigenvalues of the monodromy.

Note here there is a change of notation as compared to (2.2), ℓ becomes L . This is because we request the B_i to be Jordan blocks, which is achieved after a constant gauge transformation starting from the B_i and $Q_i(z)$ as in (2.2). So in fact $L \geq \ell$.

The last part of the corollary follows from [Del70, Thm. II.1.17].

3. REMINDER ON O-MINIMAL STRUCTURES

Recall that an o-minimal structure $(\mathcal{A}_n)_n$ consists of a set of subsets A_n of $\mathbb{R}^n$ which satisfy certain properties, in particular they comprise the semi-algebraic sets and $\mathcal{A}_1$ consists of the finite unions of intervals. In this note if not explicitly mentioned otherwise we use the o-minimal structure $\mathbb{R}_{\text{an}, \text{exp}}$ which is the structure generated by the semi-algebraic sets, the graphs of real analytic functions on $[0, 1]^n$ and the graph of $\exp: \mathbb{R} \rightarrow \mathbb{R}$. The key property of o-minimal structures that we use is that a real analytic definable function $f: (0, 1) \rightarrow \mathbb{R}$ which is not identically zero has only a finite number of zeros. A basic reference for o-minimal structures is [vD98].

Recall that a complex algebraic variety X has a canonical $\mathbb{R}_{\text{an}, \text{exp}}$ -structure (in fact a semi-algebraic structure), see [BBT23].

By an *overconvergent flat bundle* $(\mathcal{E}, \nabla)$ on $U(\rho) = (\Delta(\rho)^\times)^n \times \Delta(\rho)^{d-n}$ meromorphic along $D(\rho) = \Delta(\rho)^d \setminus U(\rho)$ we mean the restriction to $U(\rho)$ of a flat bundle on $U(\tilde{\rho})$ for some $\tilde{\rho} > \rho$ with a meromorphic structure along $D(\tilde{\rho})$ in the sense of [Del70, II.2.13], i.e. it is provided with a prolongation to a coherent sheaf of $\mathcal{M}_{U(\tilde{\rho}), D(\tilde{\rho})}$ -modules with flat connection, where $\mathcal{M}_{U(\tilde{\rho}), D(\tilde{\rho})}$ is the sheaf of rings of holomorphic functions on $U(\tilde{\rho})$ which are meromorphic along $D(\tilde{\rho})$. Such a $\mathcal{E}$ has a canonical definable structure ([BM23, Intro.]).

In a key step in our argument we use an o-minimal expansion $\mathbb{R}_{\text{sum}, \text{exp}}$ of the structure $\mathbb{R}_{\text{an}, \text{exp}}$ to a structure comprising the graphs of the ‘multisums’ of Definition 2.2. The o-minimality of this structure is shown in [vDS00], where also ‘multisums’ in several variables are discussed.

The elementary technical observation about oscillating functions that we need in the proof of our main theorem is the following. For a complex number z we write $z = \text{Re}(z) + i\text{Im}(z)$ with $\text{Re}(z), \text{Im}(z) \in \mathbb{R}$.

Lemma 3.1. *Consider $Q(z) = q_k z^{-k} + q_{k-1} z^{-k+1} + \dots + q_1 z^{-1} \in \frac{1}{z} \mathbb{C}[\frac{1}{z}]$, $b \in \mathbb{C}$, $\rho > 0$ and $d \in S^1$. If the function*

$$f: D_d(\rho) \rightarrow \mathbb{C}, \quad z \mapsto \exp(i \text{Im}(Q(z)) + b \log z)$$

is definable, where $\log z$ is some branch of the logarithm, then $Q(D_d(\rho)) \subset \mathbb{R}$ and $b \in \mathbb{R}$. In particular, if $q_k \neq 0$ then $\text{Im}(q_k d^{-k}) = 0$.

Proof. If $Q(z) \notin \mathbb{R}$ for some $z \in D_d(\rho)$ then

$$\lim_{\substack{z \in D_d(\rho) \\ z \rightarrow 0}} |\text{Im } Q(z)| = \infty$$

and grows faster than $|\log(z)|$. Thus

$$\lim_{\substack{z \in D_d(\rho) \\ z \rightarrow 0}} |\text{Im}(Q(z) + b \log z)| = \infty.$$

As the real valued function $\text{Im}(Q(z) + b \log z)$ is continuous, by the intermediate function theorem it must take all the values $2\pi M$ for $M \geq M_0, M \in \mathbb{Z}$ or $M \leq M_0, M \in \mathbb{Z}$ for some M_0 . We conclude that $f(z) = 1$ for infinitely many $z \in D_d(\rho)$, thus f oscillates. This contradicts definability.

Thus $Q(z) \in \mathbb{R}$ and in the formula for f we can then omit Q and the same argument shows that in view of

$$\lim_{\substack{z \in D_d(\rho) \\ z \rightarrow 0}} |\text{Re}(\log z)| = \infty$$

and $\text{Im}(\log z)$ constant the imaginary part of b has to vanish. $\square$

In the opposite direction we use the following simple observation.

Lemma 3.2. *Let B be a complex $n \times n$ -matrix with real eigenvalues. Then on any open sector $S \subset \mathbb{C}^\times$ any holomorphic branch of the function $z \mapsto z^B$ is definable.*

Proof. Up to replacing B by a conjugate matrix and correspondingly z^B by the same conjugate, the Jordan–Chevalley decomposition tells us that $B = D + N$, where $DN = ND$, D is real and diagonal and N is nilpotent. Then

$$z^B = \exp(D \log z) \exp(N \log z),$$

where the factor $\exp(N \log z)$ has entries which are polynomials in $\log z = \log |z| + i \arg(z)$ and where $\exp(D \log z)$ can be expressed in terms of the real exponential and logarithm functions and an analytic function in $\arg(z)$. Thus z^B is definable. $\square$

4. REDUCTION TO SECTORS

In this section we reduce the proof of Theorem 1.1 to a local statement. Based on this reduction we then recall the proof of the ‘if’ part due to Bakker–Mullane [BM23]. For the ‘only if’ part we reduce further to a local statement in one variable.

Let $\overline{X}$ be a smooth compactification of X with $\overline{X} \setminus X$ a normal crossings divisor. Let $\phi: \Delta(\rho)^d \rightarrow \overline{X}$ be a holomorphic chart with $\phi^{-1}(X) = (\Delta(\rho)^\times)^n \times \Delta(\rho)^{d-n}$. Assume that ϕ is overconvergent, i.e. extends to a holomorphic map defined on $\Delta(\rho + \epsilon)^d$ for some $\epsilon > 0$. Let $S = S_1 \times \cdots \times S_n \times \Delta(\rho)^{n-d}$ be an open, simply connected polysector in $\Delta(\rho)^d$.

Lemma 4.1. *The algebraic flat bundle $(\mathcal{E}, \nabla)$ has definable flat sections if and only if for any polysector S as above, the flat sections of $\mathcal{E}|_S$ are definable.*

Proof. We only have to show “ $\Leftarrow$ ”. Let $U \subset X$ be an arbitrary definable open subset and $\sigma: U \rightarrow \mathcal{E}|_U$ a flat section. Let $\bar{U} \subset \bar{X}$ be the closure of U . Then there are finitely many charts ϕ as above covering $\bar{U}$ and finitely many polysectors S in these charts which cover U . By definability of U , for any such polysector S , the number of connected components of $U \cap S$ is finite [vD98, Sec. 2.2]. Let S° be such a connected component. We can extend the flat section $\sigma|_{S^\circ}$ to a flat section of $\mathcal{E}|_S$ which is definable by assumption. So also $\sigma|_{S^\circ}$ is definable. The definability of σ on each component S° of $U \cap S$ implies the definability of $\sigma|_{U \cap S}$. $\square$

In the following $(\mathcal{E}, \nabla)$ denotes an overconvergent flat bundle on $U = (\Delta(\rho)^\times)^n \times \Delta(\rho)^{d-n}$ which is meromorphic along $D = \Delta(\rho)^d \setminus U$ with its canonical o-minimal structure, see Section 3. Let $S = S_1 \times \cdots \times S_n \times \Delta^{d-n}$ be an open, simply connected polysector in $\Delta(\rho)^d$ and let $\mathcal{Y} = (Y_1, \dots, Y_r)$ be a basis of the flat sections over S . Let $\mathcal{Y}M_j$ with $1 \leq j \leq n$ and with $M_i \in \mathrm{GL}_r(\mathbb{C})$ be the analytic continuation of $\mathcal{Y}$ along a simple loop around the j -th punctured disc. The $M_1, \dots, M_n$ are commuting monodromy matrices.

Theorem 4.2. *The flat sections $\mathcal{Y}$ are definable for any S as above if and only if the flat connection ∇ is regular singular and all eigenvalues of $M_1, \dots, M_n$ are unitary.*

Proof. “ $\Leftarrow$ ” (Bakker–Mullane) By assumption the eigenvalues of $M_1, \dots, M_n$ are unitary. Let $B_1, \dots, B_n$ be commuting complex matrices with real eigenvalues and $M_j = e^{2\pi i B_j}$. Then the entries of

$$\tilde{\mathcal{Y}}(z) = \mathcal{Y}(z) z_1^{-B_1} \cdots z_n^{-B_n}$$

are single valued holomorphic functions on $(\Delta(\rho)^\times)^n \times \Delta(\rho)^{d-n}$ with moderate growth, so they are meromorphic, in particular definable. As also $z_j^{-B_j}$ is definable by Lemma 3.2, so is $\mathcal{Y}$.

“ $\Rightarrow$ ” Regular singular is checked and monodromy is calculated locally around a non-singular point of D , see [Del70, Thm. II.4.1], so we can assume without loss of generality that $n = 1$. With $\tilde{\mathcal{Y}}(z) = \mathcal{Y}(z) z_1^{-B_1}$ as above we have to check that

$$\tilde{\mathcal{Y}}(z) = \sum_{j \in \mathbb{Z}} z_1^j f_j(z_2, \dots, z_d)$$

is meromorphic along D , i.e. $f_j \equiv 0$ for $j \ll 0$. For this define $E_j = \{f_j = 0\} \subset \Delta(\rho)^{d-1}$ if $f_j \not\equiv 0$ and $E_j = \emptyset$ else. This is a meager subset of $\Delta(\rho)^{d-1}$. So the countable union $\cup_j E_j$ is a meager subset of $\Delta(\rho)^{d-1}$ as well. Thus there exists $(\tilde{z}_2, \dots, \tilde{z}_d) \in \Delta(\rho)^{d-1} \setminus E$. Then if $\tilde{\mathcal{Y}}(z)$ is not meromorphic along D , the one variable function $z_1 \mapsto \tilde{\mathcal{Y}}(z_1, \tilde{z}_2, \dots, \tilde{z}_d)$ is not meromorphic either. This reduces us to the case

$d = 1$ for checking regularity. The monodromy M_1 is also calculated by restriction to these one-dimensional discs. So the implication follows from Proposition 5.1 $\square$

5. LOCAL ONE VARIABLE CASE

Our key ingredient in analyzing the local one variable case is the theory of irregular singular meromorphic differential equations and the o-minimal expansion $\mathbb{R}_{\text{sum,exp}}$ of $\mathbb{R}_{\text{an,exp}}$ involving the multisummable functions due to [vDS00].

In this section $(\mathcal{E}, \nabla)$ denotes an overconvergent flat bundle on $\Delta(\rho)^\times$ meromorphic along 0. By $d \in S^1$ we denote a direction and by $D_d(\rho) := \{xd \mid x \in (0, \rho)\} \subset \Delta(\rho)^\times$ its associated ray.

Proposition 5.1. *If there are infinitely many directions d such that all flat sections of ∇ over $D_d(\rho)$ are definable, then ∇ is regular singular with unitary monodromy eigenvalues.*

The key input in the proof is the formal and asymptotic solution theory for irregular singular differential equations as summarized in Corollary 2.5.

Proof of Proposition 5.1. The coherent sheaf of $\mathcal{M}_{(\Delta(\rho)^\times, 0)}$ -modules $\mathcal{E}$ on $\Delta(\rho)$ is locally free around 0, so without loss of generality $\mathcal{E}$ is the free sheaf $\mathcal{M}_{(\Delta(\rho), 0)}^r$. Then $(\mathcal{E}, \nabla)$ corresponds to an overconvergent meromorphic differential equation (2.1), see [Del70, Sec. I.3].

Assume Proposition 5.1 is false, and say the connection is irregular singular. We can take the pullback along $z \mapsto z^m$ for large m and perform a meromorphic gauge transformation so that we are in the situation of Corollary 2.5. Assume without loss of generality that the $Q_1(z)$ in this corollary is non-zero, say

$$Q_1(z) = q_k z^{-k} + q_{k-1} z^{-k+1} + \cdots + q_1 z^{-1}$$

with $q_k \neq 0$. Choose a direction d such that any solution of (2.1) is $\mathbb{R}_{\text{an,exp}}$ -definable on $D_d(\rho)$, such that we have $\text{Im}(q_k d^{-k}) \neq 0$ and such that the $P_d(z)$ as in Corollary 2.5 exists.

Then the function $P_d(z)$ on $D_d(\rho)$ is definable in the o-minimal structure $\mathbb{R}_{\text{sum,exp}}$, see Section 3, and so is

$$z \in D_d(\rho) \mapsto P_d^{-1}(z) \mathcal{Y}(z).$$

But multiplying the upper left entry of this matrix function with the $\mathbb{R}_{\text{an,exp}}$ -definable function

$$D_d(\rho) \rightarrow \mathbb{R}, \quad z \mapsto \exp(-\text{Re}(Q_1(z) + b_1 \log(z)))$$

we obtain the $\mathbb{R}_{\text{sum,exp}}$ -definable function

$$(5.1) \quad D_d(\rho) \rightarrow \mathbb{C}, \quad z \mapsto \exp(i \text{Im}(Q_1(z) + b_1 \log(z))).$$

However, this function is not definable in any o-minimal structure by Lemma 3.1, which is a contradiction. We conclude that all $Q_1(z), \dots, Q_L(z)$ vanish.

If the monodromy eigenvalues are not unitary, we have without loss of generality that $\text{Im}(b_1)$ does not vanish by the final part of Corollary 2.5 and still the function (5.1) would be non-definable by Lemma 3.1, which produces the same kind of contradiction. We conclude that all $b_1, \dots, b_L$ are real. $\square$

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