BOUNDING CRYSTALLINE TORSION FROM ÉTALE TORSION

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To Gérard Laumon with admiration

Abstract. In this note, we prove that given a smooth proper family over a p-adic ring of integers, one gets a control of its crystalline torsion in terms of its étale torsion, the cohomological degree, and the ramification. Our technical core result is a boundedness result concerning annihilator ideals of u^∞-torsion in Breuil–Kisin prismatic cohomology, which might be of independent interest.

Contents

1. Introduction

Let p be a prime. Let $\mathcal{O}_K$ be a complete DVR of mixed characteristic $(0, p)$ with perfect residue field k, and let C be the completion of an algebraic closure of K. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$, we have the geometric rigid-analytic generic fiber $\mathcal{X}_C$. We are interested in the interplay between two torsion phenomena associated with $\mathcal{X}$: the étale torsion $\mathrm{H}^i_{ét}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}}$ and the crystalline torsion $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$. In Bhatt–Morrow–Scholze's paper [BMS18], one learns the following:

Theorem 1.1 ([BMS18, Theorem 1.1.(ii)])**.** *There is an inequality*

$$\mathrm{length}\left(\mathrm{H}^i_{ét}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}}\right) \leq \mathrm{length}\left(\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}\right).$$

In particular, if $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}} = 0$*, then* $\mathrm{H}^i_{ét}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}} = 0$*.*

It is natural to wonder about the converse question: if $\mathrm{H}^i_{ét}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}} = 0$, then what can we say about $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$? The Künneth formula and examples in [BMS18, Section 2] shows that one cannot get any bound on the length of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$. In this paper we show that one can get a bound of the exponent of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$, defined in the Theorem below.

Theorem 1.2 (= Theorem 4.9)**.** *There is a constant $c(e, i)$ depending only on the ramification index $e = v_K(p)$ and the cohomological degree $i > 0$, such that there is an inequality*

$$\exp(\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)_{\mathrm{tors}}) \leq \exp(\mathrm{H}^i_{ét}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}}) + c(e, i).$$

Here $\exp(M)$ is defined as the smallest natural number m such that $p^m \cdot M = 0$.

Our technical tool is a generalization of some results in [LL23], concerning the annihilator ideal of u^∞-torsion in the prismatic cohomology of $\mathcal{X}$. This is the content of our Section 2. In Section 4, we give some applications of the bound of aforementioned annihilator ideals, and end with a proof of the above theorem.

Notations and Conventions. Let k be a perfect field of characteristic p, let $W = W(k)$ be its (p-typical) Witt ring. Denote $\mathfrak{S} \coloneqq W[\![u]\!]$ equipped with (u, p)-adically continuous Frobenius $\varphi\colon \mathfrak{S} \to \mathfrak{S}$ such that $\varphi|_W$ is the usual Witt vector Frobenius and $\varphi(u) = u^p$. Lastly let $E(u) \in \mathfrak{S}$ be an Eisenstein polynomial of degree e.

2. Some commutative algebra arguments

Throughout this section, we shall consider the following situation.

Situation 2.1. Let $J \subset \mathfrak{S}$ be an ideal and let $j \in \mathbb{N}$, satisfying

1. the ideal J is cofinite, namely $(p,u)^N \subset J$ for some N; and
2. we have a containment relation $E^j \cdot J \subset \varphi(J) \cdot \mathfrak{S}$.

In this situation, let σ and ρ be *defined* by the following equalities:

$$J + (p) = (p, u^\sigma) \text{ and } J + (u) = (u, p^\rho).$$

It is easy to see that $\sigma \le \lfloor \frac{e \cdot j}{p-1} \rfloor$, see for instance the proof of [LL23, Corollary 3.4].

The aim of this section is to give an explicit estimate of ρ in terms of e and j.

2.1. **Argument one.**

In this subsection, we present the first argument.

Notation 2.2. For any pair of natural numbers $(a,b) \in \mathbb{N}^2$, we let $c(a,b) := \min\{c \in \mathbb{N} \mid p^c \in (u^a, E^b)\}$.

Lemma 2.3. *Assume that $(a,b) \neq (0,0)$, then we have an inequality $c(a,b) \le \lceil \frac{a}{e} \rceil + b - 1$.*

Proof. If one of the a and b is 0, then $c = 0$, and the inequality trivially holds true as $(a,b) \neq (0,0)$. Therefore let us assume that both a and b are positive. Recall that $p \cdot \mathrm{unit} = E - u^e$ in $\mathfrak{S}$. We claim that $p^{x+y-1} \in (u^{ex}, E^y)$ for any pair of positive integers (x,y). Our claim follows from the general fact that given any pair of elements (f,g) in a ring R, we must have $(f+g)^{x+y-1} \in (f^x, g^y)$: Applying this fact to the elements $f := E$ and $g := u^e$ yields the claim. □

Lemma 2.4. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Suppose that $J + (u^a) \subset (u^a, p^N)$, then $J \subset (u^A, p^{\max(0, N - c(A,j))})$ for any natural number $A \le pa$.*

Proof. In the ring $R := \mathfrak{S}/u^A$, we have

$$p^{c(A,j)} \cdot JR \subset E^j \cdot JR \subset \varphi(J) \cdot R \subset (p^N).$$

Our claim follows from the fact that the sequence (u,p) is $\mathfrak{S}$-regular. □

Proposition 2.5. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Let $a_0, a_1, \ldots, a_n$ be a sequence of integers satisfying*

(1) $a_0 = 1$;
(2) $a_i \le p \cdot a_{i-1}$;
(3) $a_n > \frac{e \cdot j}{p-1}$.

Then $\rho \le \sum_{i=1}^n c(a_i, j)$.

In particular, if $e \cdot j < p^n(p-1)$, then we may choose $a_i = p^i$ for $i \le n-1$ and $a_n = \lfloor \frac{e \cdot j}{p-1} \rfloor + 1$, hence $\rho \le \sum_{i=1}^{n-1} \lfloor \frac{p^i}{e} \rfloor + \lfloor \frac{\lfloor \frac{e \cdot j}{p-1} \rfloor + 1}{e} \rfloor + nj$.

Proof. The second sentence follows from the first one and Lemma 2.3 as $\lceil x \rceil - 1 < x$. To see the first sentence: Let $J + (u^{a_0}) = J + (u) = (u, p^\rho)$, and assume to the contrary that $\rho > \sum_{i=1}^n c(a_i, j)$. Then applying Lemma 2.4, we see that $J + (u^{a_1}) \subset (u^{a_1}, p^{\rho - c(a_1,j)})$. Applying Lemma 2.4 again, we see that $J + (u^{a_2}) \subset (u^{a_2}, p^{\rho - c(a_1,j) - c(a_2,j)})$. Repeating the above, we finally see that $J + (u^{a_n}) \subset (u^{a_n}, p^{>0})$. But this contradicts the fact that $J + (p) = (p, u^\sigma)$ with $\sigma \le \frac{e \cdot j}{p-1} < a_n$. □

2.2. **Argument two.**

In this subsection, we present the second argument. Throughout the subsection, let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1.

Lemma 2.6. *Let $r \in [0, \infty)$ be a real number, the following map*

$$v_r \colon \mathfrak{S} \setminus \{0\} \to \mathbb{R},\ v_r(\sum_i a_i u^i) := \min\{\mathrm{ord}_p(a_i) + i \cdot r\}$$

defines an additive valuation.

Proof. It is easy to check that minimum is always attained, one can check the triangle inequality

$$v_r\left((\sum_i a_iu^i)+(\sum_i b_iu^i)\right) \geq \min\left(v_r(\sum_i a_iu^i), v_r(\sum_i b_iu^i)\right)$$

using the definition. Lastly we need to check multiplicativity:

$$v_r\left((\sum_i a_iu^i)\cdot(\sum_i b_iu^i)\right) = v_r(\sum_i a_iu^i)+v_r(\sum_i b_iu^i).$$

One checks directly that the multiplicativity holds true if one of the power series is just a monomial. Now let $\alpha := \min\{i \in \mathbb{N} \mid \mathrm{ord}_p(a_i) + i\cdot r = v_r(\sum_i a_iu^i)\}$ and $\beta := \min\{i \in \mathbb{N} \mid \mathrm{ord}_p(b_i) + i\cdot r = v_r(\sum_i b_iu^i)\}$. Using the definition, one checks that

$$v_r\left((\sum_{i\geq\alpha} a_iu^i)\cdot(\sum_{i\geq\beta} b_iu^i)\right) = v_r(\sum_i a_iu^i)+v_r(\sum_i b_iu^i).$$

Finally, by combining

- the case of one of the power series being a monomial;
- the decompositions $\sum_i a_iu^i = \sum_{i<\alpha} a_iu^i + \sum_{i\geq\alpha} a_iu^i$ and $\sum_i b_iu^i = \sum_{i<\beta} b_iu^i + \sum_{i\geq\beta} b_iu^i$ of the two power series;
- the above equality; and
- the triangle inequality,

one arrives at the multiplicativity statement which finishes the proof. □

Remark 2.7. Let us explain another perspective on the valuation v_r for $r > 0$: In Huber's adic space formalism, we may view v_r as a rank 1 point on the open unit disc, giving rise to a norm on $\mathfrak{S}[1/p]$ whose restriction to $\mathfrak{S}$ is bounded by 1. Indeed, one may view the ring $\mathfrak{S}$ as the analytic functions bounded by 1 on the open unit disc $\mathbb{D}^{\circ}_{W[1/p]}$, then the valuation v_r corresponds to the Gauss norm on the radius p^{-r} disc (the absolute value is normalized so that $|p| = p^{-1}$). Notice that for $r > 0$, we can take a rational number $s \in (0, r]$, so the said Gauss norm is a rank 1 point on the closed disc of radius p^{-s} around 0.

Notations 2.8. For any co-finite ideal $I \subset \mathfrak{S}$, let $f_I(r) := v_r(I)$, viewed as a function $f_I\colon [0,\infty) \to \mathbb{R}_{\geq 0}$. Let $I^{\mathrm{mon}} :=$ the ideal generated by $\{a_iu^i \mid \sum_i a_iu^i \in I\}$.

Namely for every power series in I, we extract out all of its monomial terms, then we use all these monomial terms of all elements in I to generate a (most likely larger) ideal. Note that I^{mon} is generated by finitely many monomial terms as $\mathfrak{S}$ is Noetherian.

Lemma 2.9. *Let $I \subset \mathfrak{S}$ be a co-finite ideal, we have natural numbers σ and ρ satisfying $I + (p) = (p, u^{\sigma})$ and $I + (u) = (u, p^{\rho})$. Then the function f_I satisfies the following:*

(1) We have an equality $f_I = f_{I^{\mathrm{mon}}}$;
(2) The function f_I is concave and continuous;
(3) The function f_I is piecewise linear, on each interval it is given by $a\cdot r + b$ with both a and b natural numbers;
(4) There exists an $\epsilon > 0$, such that

$$f_I(r) = \begin{cases} \sigma\cdot r, & r \in [0,\epsilon], \\ \rho, & r \in [1/\epsilon, \infty). \end{cases}$$

(5) We have an equality $f_{\varphi(I)}(r) = f_I(p\cdot r)$.

Proof. (1) and (5) follows from the definition of v_r. Our assumption implies that

$$I^{\mathrm{mon}} = (p^{\rho}, a_1\cdot u, a_2\cdot u^2, \ldots, a_{\sigma-1}u^{\sigma-1}, u^{\sigma}),$$

where $\mathrm{ord}_p(a_i) > 0$ (and a_i is allowed to be 0). For each of the generators above, if we look at their v_r as a function in r, we simply get a linear function with a natural number slope. The function $f_I = f_{I^{\mathrm{mon}}}$ is minimum of the above collection of linear functions, this immediately gives us (2) and (3). Using (1) and the definition of v_r, we also see that $v_r(I) = v_r(u^\sigma)$ if r is sufficiently near 0 and $v_r(I) = v_r(p^\rho)$ if $r \gg 0$, which proves (4). □

Lemma 2.10. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. We have*

(1) the function $g(r) \coloneqq v_r(E^j) = \min\Big((e \cdot j) \cdot r, j\Big)$; and

(2) an inequality $f_J(p \cdot r) \le f_J(r) + g(r)$.

Proof. (1) easily follows from our assumption on the degree e Eisenstein polynomial E. (2) follows from the assumption $E^j \cdot J \subset \varphi(J) \cdot \mathfrak{S}$ and Lemma 2.9.(5). □

Lemma 2.11. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Define a piecewise linear function*

$$h(r) = \begin{cases} \sigma \cdot r, & r \in [0, \frac{p \cdot j}{\sigma \cdot (p-1)}] \\ \frac{\sigma}{p} \cdot r + j, & r \in [\frac{p \cdot j}{\sigma \cdot (p-1)}, \frac{p^2 \cdot j}{\sigma \cdot (p-1)}] \\ \frac{\sigma}{p^2} \cdot r + 2 \cdot j, & r \in [\frac{p^2 \cdot j}{\sigma \cdot (p-1)}, \frac{p^3 \cdot j}{\sigma \cdot (p-1)}] \\ \cdots & \\ \frac{\sigma}{p^n} \cdot r + n \cdot j, & r \in [\frac{p^n \cdot j}{\sigma \cdot (p-1)}, \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}] \\ \cdots & \end{cases}.$$

Then we have $f_J(r) \le h(r)$.

We leave it to the readers to check that the $h(r)$ above is continuous, concave and increasing.

Proof. Let us check inductively on each interval that $f_J(r) \le h(r)$. For the first interval $[0, \frac{p \cdot j}{\sigma \cdot (p-1)}]$, we need to show $f_J(r) \le \sigma \cdot r$, this follows from Lemma 2.9.(2)-(4). Now we prove the induction step, so we assume that $f_J(x) \le h(x)$ whenever $x \in [0, \frac{p^n \cdot j}{\sigma \cdot (p-1)}]$ and let $r \in [\frac{p^n \cdot j}{\sigma \cdot (p-1)}, \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}]$. Our assumption on r implies that $f_J(\frac{r}{p}) \le h(\frac{r}{p}) = \frac{\sigma}{p^{n-1}} \cdot \frac{r}{p} + (n-1) \cdot j$. By Lemma 2.10, we see that

$$f_J(r) \le f_J(\frac{r}{p}) + j \le \frac{\sigma}{p^{n-1}} \cdot \frac{r}{p} + (n-1) \cdot j + j = \frac{\sigma}{p^n} \cdot r + n \cdot j = h(r).$$

□

Lemma 2.12. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Then $f_J(r) = \rho$ whenever $r \ge \frac{p \cdot j}{p-1}$.*

Proof. Let us denote by $f'_J(r)$ the left derivative of $f_J(r)$, this is a piecewise constant, decreasing, eventually 0 function, which takes values in natural numbers, thanks to Lemma 2.9.(2)-(4). Therefore all we need to show is that $f'_J(r) = 0$ for $r > \frac{p \cdot j}{p-1}$. Now Lemma 2.10 implies that $f'_J(r) \cdot (r - \frac{r}{p}) \le f_J(r) - f_J(\frac{r}{p}) \le j$. Therefore $f'_J(r) < 1$ and is a natural number, hence must be 0. □

Proposition 2.13. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. If $e \cdot j \le p^n(p-1)$, then we have $\rho \le (\frac{\sigma}{p^{n-1}(p-1)} + n) \cdot j \le (\frac{\lfloor \frac{e \cdot j}{p-1} \rfloor}{p^{n-1}(p-1)} + n) \cdot j$.*

Proof. By Lemma 2.12, we have $\rho = f_J(\frac{p \cdot j}{p-1})$. Since $\sigma \le \lfloor \frac{e \cdot j}{p-1} \rfloor \le p^n$, we see that $\frac{p \cdot j}{p-1} \le \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}$ (and we only need to prove the first inequality). Now by Lemma 2.11, we have

$$\rho = f_J(\frac{p \cdot j}{p-1}) \le h(\frac{p \cdot j}{p-1}) \le \frac{\sigma}{p^n} \cdot \frac{p \cdot j}{p-1} + n \cdot j = (\frac{\sigma}{p^{n-1}(p-1)} + n) \cdot j.$$

□

2.3. **Conclusions.** Let us first extract a concrete estimate of ρ in a special case.

Proposition 2.14. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1, with $j = 1$, and let $n \in \mathbb{N}$.*

(1) If $p \neq 2$ and $e < p^n(p-1)$, then $\rho \leq n$.
(2) If $p = 2$ and $e < 2^n$, then $\rho \leq n+1$.

Note that when $e \leq p-1$, our statement follows from the proof of [LL23, Proposition 3.5]. So in the proof below, we always assume further that $e > p-1$, in particular $n \geq 1$.

Proof. First let us assume that $p \neq 2$. Suppose that $e < p^{n-1}(p-1)^2$, then by Proposition 2.13, we see that the integer $\rho < n+1$, therefore $\rho \leq n$. If $p^{n-1}(p-1)^2 \leq e < p^n(p-1)$, then

- we have $\lfloor \frac{p^i}{e} \rfloor = 0$ for all $0 \leq i \leq n-1$ as $p \neq 2$;
- similarly $\lfloor \frac{\lfloor \frac{e}{p-1} \rfloor + 1}{e} \rfloor \leq \lfloor \frac{1}{p-1} + \frac{1}{e} \rfloor = 0$, as we have assumed that $e > p-1$.

Therefore by Proposition 2.5, we have that $\rho \leq n$.

Now in case $p = 2$, the relevant formulas simplify. When $2^{n-1} < e < 2^n$, we get $\rho \leq n+1$ by Proposition 2.5. When $e = 2^{n-1}$, we get $\rho \leq n+1$ by Proposition 2.13. ☐

Let us summarize the outcome of the previous two subsections.

Notation 2.15. For each pair of positive integers (e, j), we denote

$$d(e,j) \coloneqq \min\left(\sum_{i=1}^{n-1} \lfloor \frac{p^i}{e} \rfloor + \lfloor \frac{\lfloor \frac{e \cdot j}{p-1} \rfloor + 1}{e} \rfloor + nj, \left(\frac{\lfloor \frac{e \cdot j}{p-1} \rfloor}{p^{n-1}(p-1)} + n\right) \cdot j\right),$$

where n is the smallest natural number such that $e \cdot j < p^n(p-1)$.

Proposition 2.16. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Then we have $\rho \leq d(e,j)$.*

Proof. Combine Proposition 2.5 and Proposition 2.13. ☐

2.4. **Argument for boundedness.** Lastly let us show that an additional condition gives rise to boundedness of length of $\mathfrak{S}/J$.

Proposition 2.17. *Let $J \subset \mathfrak{S}$, $j \in \mathbb{N}$, σ and ρ be as in Situation 2.1. Assume moreover that there is an $\ell \in \mathbb{N}$ such that $E^\ell \cdot \varphi(J) \subset J$. Then we claim that $p^{(\rho + \max(j,\ell)) \cdot \sigma} \in J$. In particular, the additional assumption implies that* $\mathrm{length}(\mathfrak{S}/J) \leq (\rho + \max(j,\ell)) \cdot \sigma^2$, *hence $(u,p)^{(\rho+\max(j,\ell)) \cdot \sigma^2} \subset J$.*

Proof. For any ideal $I \subset \mathfrak{S}$, we denote $(I : p) \coloneqq \{f \in \mathfrak{S} \mid p \cdot f \in I\}$. Alternatively, the ideal is defined via the following exact sequence:

$$0 \to (I : p) \to \mathfrak{S} \xrightarrow{\cdot p} \mathfrak{S}/I.$$

Since (E, p) is a regular sequence, one checks that $E \cdot (I : p) = (E \cdot I : p)$. Using the fact that φ is flat, one checks that $\varphi(I : p) = (\varphi(I) : p)$. Therefore if we let $J_0 = J$ and inductively define $J_i = (J_{i-1} : p)$ for all $i \geq 1$, then we can make the following observations:

(1) We have $\mathfrak{S}/J_n \xrightarrow[\cong]{\cdot p^n} p^n \cdot \mathfrak{S}/J$, hence $\mathfrak{S}/(J_n + (p)) \xrightarrow[\cong]{\cdot p^n} \frac{p^n \cdot \mathfrak{S}/J}{p^{n+1} \cdot \mathfrak{S}/J}$;
(2) The ideals J_n again satisfy conditions: $E^j \cdot (-) \subset \varphi(-) \cdot \mathfrak{S}$ and $E^\ell \cdot \varphi(-) \subset (-)$.

Our task is to show that $J_n = \mathfrak{S}$ when $n \geq (\rho + \max(j,\ell)) \cdot \sigma$. Letting σ_n and ρ_n be defined by $J_n + (p) = (p, u^{\sigma_n})$ and $J_n + (u) = (u, p^{\rho_n})$, it suffices to show that $\sigma_i - \sigma_{i+\rho+\max(j,\ell)} \geq 1$. Since ρ_n is non-increasing, using the observation (2) above, it suffices to prove the above with $i = 0$.

Suppose to the contrary we have $0 < \sigma_0 = \ldots = \sigma_{\rho+\max(j,\ell)}$, we need to deduce a contradiction. This assumption, together with the observation (1) above, implies that multiplication by p on $A \coloneqq \mathfrak{S}/J$ induces isomorphisms:

$$A/pA \xrightarrow[\cong]{\cdot p} pA/p^2A \xrightarrow[\cong]{\cdot p} \ldots \xrightarrow[\cong]{\cdot p} p^{\rho+\max(j,\ell)}A/p^{\rho+\max(j,\ell)+1}A.$$

Weierstrass preparation and the definition of σ implies the existence of a *degree σ monic* polynomial $f \in J$ such that $f \equiv u^\sigma \mod p$. Since (f, p) is a regular sequence, one checks that the p-adic filtration on $B \coloneqq \mathfrak{S}/f$ also satisfies $B/pB \xrightarrow[\cong]{\cdot p} pB/p^2B \xrightarrow[\cong]{\cdot p} \ldots$. Let us now look at the map $\mathfrak{S}/(f, p^{\rho+\max(j,\ell)+1}) \twoheadrightarrow \mathfrak{S}/(J, p^{\rho+\max(j,\ell)+1})$, it is an isomorphism modulo p so, by the above knowledge of p-adic filtrations on both sides, it is an isomorphism. Therefore we have $J \equiv (f) \mod p^{\rho+\max(j,\ell)+1}$. Moreover the definition of ρ implies that the constant term of f must have p-adic valuation ρ. Now our conditions imply that there exists power series $P(u), Q(u) \in W/p^{\rho+\max(j,\ell)+1}[\![u]\!]$ such that we have equalities

$$E(u)^j \cdot f = \varphi(f) \cdot P(u) \text{ and } E(u)^\ell \cdot \varphi(f) = f \cdot Q(u)$$

in $W/p^{\rho+\max(j,\ell)+1}[\![u]\!]$. Applying the next Lemma 2.18 below, note that both f and $\varphi(f)$ satisfies the assumption on g in the said lemma, we learn that both $P(u)$ and $Q(u)$ are in fact also *monic polynomials*.

Now the constant term of left hand side of both equations are nonzero in $W/p^{\rho+\max(j,\ell)+1}$. Using Lemma 2.19 below, we see that the Newton polygon of $E(u)^j \cdot f$ is the same as that of $\varphi(f) \cdot \widetilde{P}(u)$, for an arbitrary monic polynomial lift $\widetilde{P}(u) \in W[u]$ of $P(u)$. Consequently we see that there is an inclusion of sets:

$$\{p\text{-adic valuations of roots of } \varphi(f)\} \subset \{p\text{-adic valuations of roots of } f\} \cup \{1/e\}.$$

Similarly we also have an inclusion of sets:

$$\{p\text{-adic valuations of roots of } f\} \subset \{p\text{-adic valuations of roots of } \varphi(f)\} \cup \{1/e\}.$$

Since we have an equality of subsets of $\mathbb{Q}$:

$$1/p \cdot \{p\text{-adic valuations of roots of } f\} = \{p\text{-adic valuations of roots of } \varphi(f)\},$$

we arrive at the following contradiction:

$$\{p\text{-adic valuations of roots of } f\} \cup \{1/e\} = (1/p \cdot \{p\text{-adic valuations of roots of } f\}) \cup \{1/e\}.$$

Therefore we see that we cannot have $(\rho + \max(j, \ell))$ many σ's being all equal, which finishes the proof. □

In the above proof, we have summoned the following two general facts.

Lemma 2.18. *Let f and g be* monic *polynomials in $W_m[u]$, with $g \equiv u^n$ modulo p. Assume that there is an $h \in W_m[\![u]\!]$ satisfying $f = g \cdot h$ in $W_m[\![u]\!]$. Then in fact $h \in W_m[u]$ is also a* monic *polynomial.*

Proof. Let us do induction on m. The case of $m = 1$ is clear. Then by induction, we see that $h = h' + p^{m-1} \cdot h''$ where h' is a monic polynomial satisfying the equality $f \equiv g \cdot h'$ modulo p^{m-1} and $h'' \in W_m[\![u]\!]$. After subtracting the monic polynomial $g \cdot h'$ from both sides, we arrive at an equality

$$p^{m-1} f' = p^{m-1} g \cdot h'' \in W_m[\![u]\!],$$

where f' is a polynomial. Our assumption on g now guarantees that $h'' \equiv$ a polynomial modulo p. Therefore we know that h is a polynomial. Comparing the leading coefficients of the equation $f = g \cdot h$ (now in $W_m[u]$), we see that h is monic. □

Lemma 2.19. *Let f and g be monic polynomials in $W[u]$, assume that there exists a positive integer m such that $f \equiv g$ modulo p^m with constant term in the common reduced monic polynomial non-zero. Then f and g have the same Newton polygon.*

Proof. We first observe that the assumption implies:

(1) Both f and g are of the same degree d;
(2) their constant terms have the same p-adic valuation: $v_p(f(0)) = v_p(g(0)) = n < m$.

We learn that their Newton polygon must be below the line segment connecting $(0, n)$ and $(d, 0)$. Therefore, for any such polynomial and for $0 < i < d$, in order for $(i, v_p(a_i))$ to lie on the Newton polygon, we necessarily have $v_p(a_i) \leq n \cdot (1 - \frac{i}{d})$. In particular, perturbing a_i by an element whose p-adic valuation is larger than p^n does not change its Newton polygon. Since g is obtained from f via perturbing coefficients by elements divisible by p^m, we see that the Newton polygon is unchanged. □

3. Some prismatic cohomology facts

In this section, we recall some statements concerning torsion in prismatic cohomology. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$.

Remark 3.1. Recall (see [BMS18, Proposition 4.3] and [BS22, Theorem 1.8.6]) that the prismatic cohomology $\mathfrak{M}^i := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, being a Breuil–Kisin module, admits the following canonical exact sequences:

$$0 \to \mathfrak{M}^i_{\mathrm{tors}} = \mathfrak{M}^i[p^\infty] \to \mathfrak{M}^i \to \mathfrak{M}^i_{\mathrm{tf}} \to 0,$$

$$0 \to \mathfrak{M}^i_{\mathrm{tf}} \to (\mathfrak{M}^i)^{\vee\vee} \to \overline{\mathfrak{M}^i} \to 0,$$

where $(\mathfrak{M}^i)^{\vee\vee}$ is the double dual (or reflexive hull) of $\mathfrak{M}^i$ which is finite free over $\mathfrak{S}$ and $\overline{\mathfrak{M}^i}$ is supported at the closed point (p, u) of $\mathrm{Spec}(\mathfrak{S})$.

The following result is the main reason why we studied the kind of ideal J in Situation 2.1.

Proposition 3.2. *Let $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathfrak{M}^i_n := \mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})/^L p^n)$ (where $n = \infty$ means that we do not perform the reduction at all). Then we have the following:*

(1) For all $i \geq 0$, there exists maps $F \colon \varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n \to \mathfrak{M}^i_n$ and $V \colon \mathfrak{M}^i_n \to \varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n$ such that both $F \circ V$ and $V \circ F$ are the same as multiplication by E^i;

(2) For all $i > 0$, multiplication by E^{i-1} on $\varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n$ factors through a submodule of $\mathfrak{M}^i_n$.

In particular, when $i > 0$, let J be the annihilator ideal of $\mathfrak{M}^i_n[u^\infty]$. Then the ideal J and $(j, \ell) = (i-1, i)$ satisfy the conditions in Situation 2.1 and Proposition 2.17.

When $n = \infty$, statement (1) follows from [BS22, Theorem 1.8.(6)]. In general, both (1) and (2) follow from the observation made in [LL23, Proposition 3.2]. For the convenience of the readers, let us sketch the argument below.

Proof. Recall that the Frobenius-twisted prismatic cohomology admits Nygaard filtrations, see [BS22, Section 15]. In particular, for any $j \geq 0$, there are natural maps $\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \to \varphi^*_{\mathfrak{S}}\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ and $\varphi^*_{\mathfrak{S}}\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n) \to \mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n)$ such that compositions either way are the same as multiplication by E^j. Moreover these Nygaard filtrations admit divided Frobenius maps to prismatic cohomology: $\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \xrightarrow{\varphi_j} \mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$.

By [LL25, Lemma 7.8.(3)], the induced map $\mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \xrightarrow{\varphi_j} \mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ is an isomorphism. This gives (1) by considering the i-th step of the Nygaard filtration. Also by [LL25, Lemma 7.8.(3)], when $j > 0$, the induced map $\mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^{j-1}/p^n) \xrightarrow{\varphi_{j-1}} \mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ is injective. This gives (2) by considering the $(i-1)$-st step of the Nygaard filtration. The last sentence is a consequence of (1) and (2). ☐

Remark 3.3. Let us take the opportunity to correct an error in [LL25, Lemma 7.8.(3)]. The proof has a gap in its last sentence: namely, when we use the same proof strategy to run the argument for proving the derived mod p^m versions, the cohomological estimate might be off by 1 cohomological degree due to p-torsion in $\Omega^{i+1}_{X/(A/I)}$, and this p-torsion subsheaf is nonzero exactly when A/I contains p-torsion (and $X/(A/I)$ has relative dimension at least $i+1$). Therefore, by the proof strategy of loc. cit. we get the following conclusion: The statement of [LL25, Lemma 7.8.(1)-(3)] is correct as is, but for their derived mod p^m analogs, one needs an extra assumption that (A, I) is a transversal prism (namely A/I is p-torsion free). So, one just needs to change the last sentence to "Moreover their derived mod p^m counterparts hold *as long as (A, I) is transversal*." Fortunately, the Breuil–Kisin prism is an example of such, which justifies our usage of [LL25, Lemma 7.8.(3)] in the above proof. Lastly we point out that in the proof of [LL25, Lemma 7.8.(3)], the authors give a reference to [BS22, Theorem 15.2.(2)] for the cohomological estimate, but the more appropriate reference seems to be rather [BS22, Theorem 15.3].

The rest of this section concerns the A_{inf} cohomology defined in [BMS18, Theorem 1.8], let us recall some key definitions and properties below.

Notations 3.4. Let C be the completion of an algebraic closure of K, with its tilt $C^\flat$ defined as follows: Consider the ring of integers $\mathcal{O}_C \subset C$, then define $\mathcal{O}_C^\flat := \lim_\varphi(\mathcal{O}_C/p)$. Given a sequence of elements $\{x_i\}_{i\in\mathbb{N}}$ of $\mathcal{O}_C/p$ satisfying $x_i^p = x_{i-1}$, we denote by $\underline{x}$ its corresponding element in $\mathcal{O}_C^\flat$. It is a fact that $\mathcal{O}_C^\flat$ is a rank 1 valuation ring, whose fraction field $\mathrm{Frac}(\mathcal{O}_C^\flat) =: C^\flat$ is an algebraically closed complete non-archimedean field of equal characteristic p. The maximal ideal of $\mathcal{O}_C^\flat$ is given by $\mathfrak{m}_C^\flat = \{\underline{x} \in \mathcal{O}_C^\flat \mid x_0 \in \mathfrak{m}_C/(p\cdot\mathcal{O}_C) \subset \mathcal{O}_C/p\}$. For more on this, we refer readers to [Sch12, Section 3].

Fix a choice of compatible p-power primitive roots of unity $(1, \zeta_p, \zeta_{p^2}, \cdots)$, then the sequence $\{\zeta_{p^i}\}_{i\in\mathbb{N}}$ defines an element $\epsilon \in \mathcal{O}_C^\flat$. The Fontaine period ring A_{inf} is defined as the (p-typical) Witt ring of $\mathcal{O}_C^\flat$, equipped with Frobenius automorphism φ. The following two elements $\mu := [\epsilon] - 1$ and $\widetilde{\xi} = \varphi(\xi) = \frac{\varphi(\mu)}{\mu}$ in A_{inf} are important to us.

In the rest of this section C can be any algebraically closed complete non-archimedean field of mixed characteristic $(0, p)$.

Remark 3.5. Let $\mathfrak{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_C)$ with its rigid generic fiber $X := \mathfrak{X}_C$. There is a natural map of sites $\nu\colon X_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{Zar}}$, then according to [BMS18, Definition 8.1 and 9.1], one defines

$$A\Omega_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}) \text{ and } \widetilde{\Omega}_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}).$$

For the purpose of this paper, we merely view the above as objects in $D(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}})$. The A_{inf} cohomology is then defined as

$$\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := \mathrm{R}\Gamma(\mathfrak{X}_{\mathrm{Zar}}, A\Omega_{\mathfrak{X}}).$$

By [BMS18, Theorem 1.8], all cohomology groups are Breuil–Kisin–Fargues modules (see [BMS18, Definition 4.22]). Analogous to Remark 3.1, using [BMS18, Proposition 4.13], we see that $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$ also admits a natural exact sequence:

$$0 \to M^i_{\mathrm{tors}} = M^i[p^{\gg 0}] \to M^i \to M^i_{\mathrm{free}} \to \overline{M^i} \to 0,$$

with all modules appearing above, regarded as A_{inf}-complexes, perfect.

In general, (derived) reduction modulo an element certainly does not commute with $L\eta$ with respect to another element. Therefore it is surprising to learn (see [BMS18, Theorem 9.2.(1)]) that the natural map $A\Omega_{\mathfrak{X}}/\widetilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism! In [Bha18], at least if we work at the level of almost mathematics with respect to $[\mathfrak{m}_C^\flat]$, one finds a conceptual proof for this fact.

Proposition 3.6 ([Bha18, Lemma 5.16 and Proposition 7.5])**.** *The natural map $A\Omega_{\mathfrak{X}}/\widetilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}}$ is an almost, with respect to $[\mathfrak{m}_C^\flat]$, isomorphism in $D(\mathfrak{X}_{\mathrm{Zar}}, A^a_{\mathrm{inf}})$.*

Let us sketch the proof for later use.

Sketch of proof in loc. cit. The Lemma 5.16 in loc. cit. provides such a natural map, as well as a criterion for when the map is an almost isomorphism: it suffices for the cohomology sheaves of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})/\mu$ to be almost $\widetilde{\xi}$-torsionfree. Since $\widetilde{\xi} = \frac{(\mu+1)^p-1}{\mu} = \mu^{p-1} + \ldots + p\cdot\mu + p \equiv p$ modulo μ, it is equivalent to these cohomology sheaves being almost p-torsionfree. This later claim follows from Theorem 4.14 and Lemma 7.1 in loc. cit. ☐

The above admits a direct generalization.

Proposition 3.7. *Define $\widetilde{\Omega}^{(n)}_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}^n) \in D(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}})$. Then the natural map $A\Omega_{\mathfrak{X}}/\widetilde{\xi}^n \to \widetilde{\Omega}^{(n)}_{\mathfrak{X}}$ is an almost, with respect to $[\mathfrak{m}_C^\flat]$, isomorphism in $D(\mathfrak{X}_{\mathrm{Zar}}, A^a_{\mathrm{inf}})$.*

Proof. Using again [Bha18, Lemma 5.16], we are reduced to showing that the cohomology sheaves of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})/\mu$ are almost $\widetilde{\xi}^n$-torsionfree. Since this is equivalent to these sheaves being almost $\widetilde{\xi}$-torsionfree, we are done thanks to the proof of Proposition 3.6. ☐

Lemma 3.8. *Set $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$, then there exists an $N \gg 0$ such that $M^i[\widetilde{\xi}^\infty] = M^i[\widetilde{\xi}^N]$. Moreover $M^i[\widetilde{\xi}^\infty]$ is a finitely presented coherent A_{inf}-module.*

Proof. By Remark 3.5, there exists an $m \in \mathbb{N}$ such that the torsion submodule in M^i is given by $M := M^i[p^m]$, which is a perfect complex. In particular, it is finitely presented. Using [BMS18, Lemma 3.26], we know that $W_m(\mathcal{O}_C^\flat)$ is a coherent ring. By [Sta25, Tag 05CX], we see that M is a coherent $W_m(\mathcal{O}_C^\flat)$-module. Therefore we are reduced to showing: if M is a finitely presented $W_m(\mathcal{O}_C^\flat)$-module, then there exists an $N \gg 0$ such that $M[\widetilde{\xi}^\infty] = M[\widetilde{\xi}^N]$. Indeed, we may then apply [Sta25, Tag 05CW] to see that $M[\widetilde{\xi}^N] = \ker(M \xrightarrow{\widetilde{\xi}^N} M)$ is a finitely presented coherent $W_m(\mathcal{O}_C^\flat)$-module.

Let us prove the above claim, by induction on the smallest power p^m of p that annihilates M. If M is annihilated by p, this follows from the fact that $\mathcal{O}_C^\flat$ is a rank one valuation ring. Since $W_m(\mathcal{O}_C^\flat)$ is a coherent ring, we know that both $Q := M[p]$ and $M/Q \cong \mathrm{Im}(M \xrightarrow{\cdot p} M)$ are finitely presented $W_m(\mathcal{O}_C^\flat)$-modules. By induction, if $m > 1$, we see that the $\widetilde{\xi}^\infty$-torsion parts in both Q and M/Q are annihilated by $\widetilde{\xi}^{N'}$ for some $N' \gg 0$. By the snake lemma, there is a natural exact sequence

$$0 \to Q[\widetilde{\xi}^\infty] \to M[\widetilde{\xi}^\infty] \to M/Q[\widetilde{\xi}^\infty].$$

One immediately sees that $M[\widetilde{\xi}^\infty]$ is annihilated by $\widetilde{\xi}^{2N'}$, hence we are done. □

Remark 3.9. Since this paper cares about the $\mathfrak{X}$'s coming from some $\mathcal{X}/\mathcal{O}_K$ for some p-adic field K via base change, let us point out that in this particular case the above Lemma 3.8 follows from the first paragraph of the proof of Theorem 4.2 and the fact that $\mathfrak{S}$ is Noetherian.

The following is inspired by the proof of [Min21, Lemma 5.1].

Proposition 3.10. *Let $i > 0$ and set $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$, then $M^i[\widetilde{\xi}^\infty]$ is almost, with respect to $[\mathfrak{m}_C^\flat]$, annihilated by μ^{i-1}. In particular, let $J_{\mathrm{inf}} \subset A_{\mathrm{inf}}$ be the annihilator of $M^i[\widetilde{\xi}^\infty]$, then we have an inclusion $\mu^{i-1} \cdot [\mathfrak{m}_C^\flat] \subset J_{\mathrm{inf}}$.*

Proof. Let n be an arbitrary positive integer. Recall [BMS18, Corollary 6.5] that the $L\eta$ functor commutes with canonical truncation. Applying [BMS18, Lemma 6.9], we see that there is a commutative diagram in $D(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}}^a)$:

$$\begin{array}{ccc} \tau^{\leq(i-1)} A\Omega_{\mathfrak{X}} & \longrightarrow & \tau^{\leq(i-1)} \widetilde{\Omega}_{\mathfrak{X}}^{(n)} \\ f_1 \downarrow \uparrow g_1 & & f_2 \downarrow \uparrow g_2 \\ \tau^{\leq(i-1)} R\nu_*(\mathbb{A}_{\mathrm{inf},X}) & \longrightarrow & \tau^{\leq(i-1)} R\nu_*(\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}^n), \end{array}$$

where both horizontal arrows are induced by $\tau^{(i-1)}$ applied to the (derived) reduction modulo $\widetilde{\xi}^n$ map, and the composition of f_j and g_j in either direction is μ^{i-1} for $j = 1, 2$.

By [Sch13, Theorem 5.1 and proof of Theorem 8.4], we get almost isomorphisms

$$\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \cong \mathrm{R}\Gamma(X_{\text{proét}}, \mathbb{A}_{\mathrm{inf}}) \text{ and } \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/\widetilde{\xi}^n \cong \mathrm{R}\Gamma(X_{\text{proét}}, \mathbb{A}_{\mathrm{inf}}/\widetilde{\xi}^n)$$

with respect to $[\mathfrak{m}_C^\flat]$. Now we take $(i-1)$-st cohomology of the diagram above, and arrive at the following commutative diagram of almost A_{inf}-modules:

$$\begin{array}{ccc} \mathrm{H}^{i-1}_{A_{\mathrm{inf}}}(\mathfrak{X}) & \longrightarrow & \mathrm{H}^{i-1}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}^{(n)}) \cong \mathrm{H}^{i-1}(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})/\widetilde{\xi}^n) \\ f_1 \downarrow \uparrow g_1 & & f_2 \downarrow \uparrow g_2 \\ \mathrm{H}^{i-1}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} & \twoheadrightarrow & \mathrm{H}^{i-1}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/\widetilde{\xi}^n, \end{array}$$

where the identification of top-right item uses Proposition 3.7, and the composition of f_j and g_j in either direction is μ^{i-1} for $j = 1, 2$. Since the cokernel of the top arrow is, as an almost A_{inf}-module, given by $\mathrm{H}^i_{A_{\mathrm{inf}}}(\mathfrak{X})[\widetilde{\xi}^n]$, we see that $\mathrm{H}^i_{A_{\mathrm{inf}}}(\mathfrak{X})[\widetilde{\xi}^n]$ is almost annihilated by μ^{i-1}. By Lemma 3.8, we can choose n large enough so that $\mathrm{H}^i_{A_{\mathrm{inf}}}(\mathfrak{X})[\widetilde{\xi}^n] = \mathrm{H}^i_{A_{\mathrm{inf}}}(\mathfrak{X})[\widetilde{\xi}^\infty]$. □

Lemma 3.11. *Let R be a coherent ring, and let M be a finitely presented R-module. Then the annihilator ideal of M is finitely presented.*

We thank a referee for helping us to simplify the following proof.

Proof. By [Sta25, Tag 05CX], the module M is coherent. Choose generators $\{x_i; 1 \leq i \leq n\} \subset M$, and write the annihilator ideal as $Ann(M) = \ker(R \xrightarrow{(\cdot x_i)_{1\leq i\leq n}} \prod_{i=1}^n M)$. Applying [Sta25, Tag 05CW] to this map implies that the annihilator ideal is finitely generated. Therefore we are done as R is coherent. $\square$

Corollary 3.12. *With setup and notation as in Proposition 3.10. The ideal $J_{\inf} \subset A_{\inf}$ is a finitely generated ideal containing some power of p, and we have $\mu^{i-1} \in J_{\inf}$.*

Proof. By Lemma 3.8 and its proof, we see that $M^i[\widetilde{\xi}^\infty]$ is a finitely presented $W_m(\mathcal{O}_C^\flat)$-module for some m, and the ideal $J_{\inf}$ is the preimage under the projection $A_{\inf} \xrightarrow{\bmod p^m} W_m(\mathcal{O}_C^\flat)$ of the annihilator ideal $J' \subset W_m(\mathcal{O}_C^\flat)$ of $M^i[\widetilde{\xi}^\infty]$. From this we already know that $p^m \in J_{\inf}$. For the finite generation claim, it suffices to know that J' is finitely presented, which follows from combining Lemma 3.8 and Lemma 3.11.

It remains to show that $\mu^{i-1} \in J'$, which is equivalent to $W_m(\mathcal{O}_C^\flat) = \ker(W_m(\mathcal{O}_C^\flat) \xrightarrow{\cdot \mu^{i-1}} W_m(\mathcal{O}_C^\flat)/J')$. Using [Sta25, Tag 05CW] we see that the kernel is a finitely generated ideal. By Proposition 3.10, we see this finitely generated ideal contains the image of $[\mathfrak{m}_C^\flat]$, therefore it must be the unit ideal. $\square$

4. Applications

Throughout this section, let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. In this section, we deduce consequences of the previous sections. We begin with an auxilliary lemma.

Lemma 4.1. *Let C be a complete algebraically closed nonarchimedean extension of $\mathbb{Q}_p$. Let $v_{C^\flat}$ be the valuation on the tilt $C^\flat$, normalized so that $v_{C^\flat}(p^\flat) = 1$. Let $j > 0$ and consider the Teichmüller expansion*

$$\mu^j = \sum_{i\geq 0} p^i \cdot [x_i^{(j)}] \in W(\mathcal{O}_C^\flat),$$

then we have $v_{C^\flat}(x_{j\ell}^{(j)}) = j \cdot \frac{p}{p^\ell (p-1)}$ for any $\ell \in \mathbb{N}$.

Proof. Recall that the addition in (p-typical) Witt vectors of a perfect ring R is defined in the following manner. First there are universal polynomials $Q_i(X, Y) \in \mathbb{Z}[X, Y]$ defined inductively by

$$X^{p^n} + Y^{p^n} = \sum_{i=0}^n p^i Q_i^{p^{n-i}}.$$

Then we have

$$[x] + [y] = \sum_{i\geq 0} p^i \cdot [Q_i(x^{1/p^i}, y^{1/p^i})] \text{ in } W(R)$$

for any $x, y \in R$. We can inductively see that

- Each $Q_i(X, Y)$ is a homogeneous degree p^i polynomial;
- $Q_0(X, Y) = X + Y$;
- whenever $i > 0$ there is an expansion of the form $Q_i(X, Y) = \sum_{1\leq m\leq p^i-1} a_m X^m Y^{p^i-m}$ with $a_1 = a_{p^i-1} = 1$.

For $x_i := x_i^{(1)}$, we have from the above two expansions

$$[\epsilon] + \sum_{i>0} p^i [Q_i((\epsilon - 1)^{1/p^i}, 1)] = [\epsilon - 1] + 1 = [\epsilon] + (-1) \cdot \sum_{i>0} p^i [x_i],$$

and $x_0 = \epsilon - 1$. In particular, we see that

$$(-1) \cdot \sum_{i>0} p^i [Q_i((\epsilon - 1)^{1/p^i}, 1)] = \sum_{i>0} p^i [x_i].$$

We claim that our lemma follows from this equality, together with the discussion of "Newton polygon" in [FF18, Subsection 1.5].

Let us first summarize necessary definitions and facts concerning Newton polygons: In [FF18, Definition 1.5.2], to any element $y = \sum_{i\geq 0} p^i \cdot [y_i] \in A_{\inf}$, the authors define $\mathcal{N}ewt(y)$ to be the function $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$

whose graph is the highest convex non-increasing polygon below the points $\{(n, v_{C^\flat}(y_n)) \mid n \in \mathbb{N}\}$. By how $\mathcal{N}ewt(y)$ is defined, we see that if (n, v_n) is a turning point of its graph, then $v_{C^\flat}(y_n) = v_n$. On [FF18, p. 20], the authors conclude that $\mathcal{N}ewt(y \cdot z) = \mathcal{N}ewt(y) * \mathcal{N}ewt(z)$, where the operation $*$ of convex functions is defined on [FF18, p. 18]. Using this, one checks that $\mathcal{N}ewt(u \cdot y) = \mathcal{N}ewt(y)$ if u is a unit.

Now we are ready to prove the claim for $j = 1$: Using the previous paragraph, we see that

$$\mathcal{N}ewt(\sum_{i>0} p^i [x_i]) = \mathcal{N}ewt(\sum_{i>0} p^i [Q_i((\epsilon - 1)^{1/p^i}, 1)]).$$

By the third observation on these Q_i's, we have $v_{C^\flat}(Q_i((\epsilon-1)^{1/p^i}, 1)) = \frac{p}{p^i(p-1)}$ for all $i > 0$. So the Newton polygon goes precisely through $\{(n, \frac{p}{p^{n-1}(p-1)}) \mid n \in \mathbb{N}\}$ for all $n \geq 1$, and these points are all turning points. In the end we deduce that $v_{C^\flat}(x_i) = \frac{p}{p^i(p-1)}$ for all $i > 0$ as well.

The $j = 1$ case implies the general case, as follows: Chasing through the definition of $*$, the graph of $\mathcal{N}ewt(y^j)$ is the original graph of $\mathcal{N}ewt(y)$ scaled by multiplication by j. Therefore the turning points of $\mathcal{N}ewt(\mu^j)$ are given by $\{(j \cdot n, j \cdot \frac{p}{p^{n-1}(p-1)}) \mid n \in \mathbb{N}\}$, finishing the proof. □

With the above preparation, we can prove the following.

Theorem 4.2. *Let $i > 0$, denote $\mathfrak{M}^i := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, and let J be the annihilator ideal of $\mathfrak{M}^i[u^\infty]$. Let ρ be defined by $J + (u) = (u, p^\rho)$. If $e \cdot (i-1) < p^n(p-1)$, then $\rho \leq (i-1) \cdot n$.*

By [LL23, Corollary 3.8 or Remark 3.9], the $\mathfrak{M}^1$ is always finite free. Therefore in the following proof, we always assume that $i \geq 2$, hence $(i-1) > 0$. So we may summon Lemma 4.1 for $j = (i-1)$.

Proof. Let $\mathfrak{X} := \mathcal{X}_{\mathcal{O}_C}$, and set $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$. After choosing compatible p-power roots of π in $\mathcal{O}_C$, we get an element $\pi^\flat \in \mathcal{O}_C^\flat$ (see Notations 3.4). We may consider the map of prisms which is p-completely faithfully flat:

$$f \colon (\mathfrak{S} = W[\![u]\!], (E)) \to (A_{\mathrm{inf}}, (\xi)),$$

with $f(u) = [\pi^\flat]$. By [BS22, Theorem 1.8.(5) and Theorem 17.2], we get a canonical isomorphism $\mathfrak{M}^i \otimes_{\mathfrak{S}, \varphi \circ f} A_{\mathrm{inf}} \cong M^i$. Using the p-completely flatness of f, together with structural results mentioned in Remark 3.1 and Remark 3.5, we also get $\mathfrak{M}^i[u^\infty] \otimes_{\mathfrak{S}, \varphi \circ f} A_{\mathrm{inf}} \cong M^i[\widetilde{\xi}^\infty]$. In particular, using again the p-completely flatness of f, the annihilator ideal J_{inf} of $M^i[\widetilde{\xi}^\infty]$ is given by $(\varphi \circ f)(J) \cdot A_{\mathrm{inf}}$.

Now suppose that $\rho > (i-1) \cdot n$, then we have $J \subset (u, p^{(i-1)\cdot n+1})$, consequently $J_{\mathrm{inf}} \subset J'_{\mathrm{inf}} := ([(\pi^\flat)^p]), p^{(i-1)\cdot n+1})$. Notice that an element $x = \sum_{m \geq 0} p^m \cdot [x_m] \in A_{\mathrm{inf}}$ lies in J'_{inf} if and only if $v_{C^\flat}(x_m) \geq \frac{p}{e}$ for all $m \leq (i-1) \cdot n$.

Corollary 3.12 says that $\mu^{i-1} \in J_{\mathrm{inf}}$. Therefore by the above paragraph, in the Teichmüller expansion of $\mu^{i-1} = \sum_{m \geq 0} p^m \cdot [x_m^{(i-1)}]$, we must have $v_{C^\flat}(x_m^{(i-1)}) \geq \frac{p}{e}$ for all $m \leq (i-1) \cdot n$. On the other hand, by Lemma 4.1 we have

$$v_{C^\flat}(x^{(i-1)}_{(i-1)\cdot n}) = (i-1) \cdot \frac{p}{p^n(p-1)}$$

contradicting the assumption $e \cdot (i-1) < p^n(p-1)$. □

Remark 4.3. Let us point out that the above proof no longer works for the cohomology of $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})/^L p^n$: Indeed in the proof above we have summoned the key result Proposition 3.10. The basic fact used to prove Proposition 3.10 is Proposition 3.7: it says that applying the operations $L\eta_\mu$ and $/^L \widetilde{\xi}^n$ to the complex $\mathrm{R}\nu_*(\mathbb{A}_{\mathrm{inf},X})$ commute at the almost level. Following Bhatt's proof (see the proof of Proposition 3.6), this in turn follows from the fact that the cohomology sheaves of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})/\mu$ are almost $\widetilde{\xi}$-torsionfree. Therefore we no longer have the (almost) commutation of the two functors applied to the complex $\mathrm{R}\nu_*(\mathbb{A}_{\mathrm{inf},X}/^L p^n)$. This in turn stops us from generalizing Proposition 3.10 to the derived modulo p^n version.

In practice, it is also important to understand the cohomology of $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})/^L p^n$. We have arguments purely from commutative algebra, at the expense of getting a slightly worse bound.

Theorem 4.4. *Let $n \in \mathbb{N} \cup \{\infty\}$ and let $i > 0$, denote $\mathfrak{M}^i_n := \mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})/^L p^n)$ (where $n = \infty$ means that we do not perform the reduction at all), and let J be the annihilator ideal of $\mathfrak{M}^i_n[u^\infty]$. Lastly, let σ and ρ be defined by $J + (p) = (p, u^\sigma)$ and $J + (u) = (u, p^\rho)$, we have*

(1) inequalities $\sigma \leq \lfloor \frac{e\cdot(i-1)}{p-1} \rfloor$ *and* $\rho \leq d(e, i-1)$*;*
(2) a belonging $p^{(\rho+i)\cdot\sigma} \in J$*; and*
(3) an inclusion $(u,p)^{(\rho+i)\cdot\sigma^2} \subset J$*.*

Proof. Using Proposition 3.2, statement (1) follows from Proposition 2.16, statement (2) follows from Proposition 2.17. As for statement (3): we have an inclusion $(u,p)^\sigma \subset (u^\sigma, p) = J + (p)$ by the definition of σ. Then we have $(J + (p))^{(\rho+i)\cdot\sigma} \subset J + (p^{(\rho+i)\cdot\sigma}) = J$ thanks to (2). □

In [LL23] one finds results relating pathologies in p-adic geometry with u-torsion in prismatic cohomology, here let us update the conclusions with our new estimates.

Proposition 4.5. *Assume that the formal scheme* $\mathcal{X}$ *has an* $\mathcal{O}_K$*-point. Let* $f\colon \mathrm{Alb}(\mathcal{X}_k) \to \mathrm{Alb}(\mathcal{X}_K)_k$ *be the natural map discussed in the beginning of* [LL23, Subsection 4.1]*. Then* $\ker(f)$ *is* p^n*-torsion if* $e < p^n(p-1)$*.*

Proof. This follows from combination of Theorem 4.2, [LL23, Proposition 4.1] and [LL23, Theorem 4.2]. □

Proposition 4.6. *Let* C *be the completion of an algebraic closure of* K*, let* $n \in \mathbb{N} \cup \{\infty\}$ *and let* $i > 0$*, consider the specialization map*

$$\mathrm{Sp}^i_n \colon \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C, \mathbb{Z}/p^n)$$

discussed in the beginning of [LL23, Subsection 4.2] *(here again* $n = \infty$ *means that we do not perform reduction at all). Then* $\ker(\mathrm{Sp}^i_n)$ *is* $p^{d(e,i-1)}$*-torsion.*

Proof. This follows from Theorem 4.4 and [LL23, Theorem 4.14]. □

From now on, we use the notation from Remark 3.1. Let us observe that one can control $\overline{\mathfrak{M}^i}$ in terms of $\mathfrak{M}^i/p^N$ for some $N \gg 0$.

Lemma 4.7. *Let* $\mathfrak{M}$ *be any finitely generated* $\mathfrak{S}$*-module admitting exact sequences as in Remark 3.1, let* p^m *be such that it annihilates both* $\mathfrak{M}_{\mathrm{tors}}$ *and* $\overline{\mathfrak{M}}$*, then there is an exact sequence:*

$$0 \to \mathfrak{M}_{\mathrm{tors}} \oplus \overline{\mathfrak{M}} \to \mathfrak{M}/p^N \to (\mathfrak{M})^{\vee\vee}/p^N \to \overline{\mathfrak{M}} \to 0,$$

for any $N \geq 2m$*. In particular, there is an identification* $\mathfrak{M}/p^N[u^\infty] \simeq \mathfrak{M}[u^\infty] \oplus \overline{\mathfrak{M}}$ *whenever* $N \geq 2m$*.*

Proof. For any natural number n, we have canonical exact sequences:

$$0 \to \mathfrak{M}_{\mathrm{tors}}/p^n \to \mathfrak{M}/p^n \to \mathfrak{M}_{\mathrm{tf}}/p^n \to 0,$$
$$0 \to \overline{\mathfrak{M}}[p^n] \to \mathfrak{M}_{\mathrm{tf}}/p^n \to (\mathfrak{M})^{\vee\vee}/p^n \to \overline{\mathfrak{M}}/p^n \to 0.$$

The second sequence implies that $\mathfrak{M}_{\mathrm{tf}}/p^n[u^\infty] \cong \overline{\mathfrak{M}}[p^n]$, with the natural transitions map from $(n+1)$-st level to n-th level on the left hand side identified with the multiplication by p map on the right hand side. Now let us denote $\mathfrak{N}_n \coloneqq \{x \in \mathfrak{M}/p^n \mid u^{\gg 0}x \in \mathfrak{M}_{\mathrm{tors}}/p^n \subset \mathfrak{M}/p^n\}$, then we have a canonical isomorphism $(\mathfrak{M}/p^n)/\mathfrak{N}_n \cong (\mathfrak{M}_{\mathrm{tf}}/p^n)/\overline{\mathfrak{M}}[p^n]$ and commutative diagrams of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}_{\mathrm{tors}}/p^{n+1} & \longrightarrow & \mathfrak{N}_{n+1} & \longrightarrow & \overline{\mathfrak{M}}[p^{n+1}] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{mod}\ p^n} & & \downarrow & & \downarrow{\scriptstyle \cdot p} & & \\ 0 & \longrightarrow & \mathfrak{M}_{\mathrm{tors}}/p^n & \longrightarrow & \mathfrak{N}_n & \longrightarrow & \overline{\mathfrak{M}}[p^n] & \longrightarrow & 0. \end{array}$$

Our choice of $N \geq 2m$ implies that the transition map $\mathfrak{M}_{\mathrm{tors}}/p^N \xrightarrow[\cong]{\mathrm{mod}\ p^m} \mathfrak{M}_{\mathrm{tors}}/p^m$ is an isomorphism, and the transition map $\overline{\mathfrak{M}}[p^N] \xrightarrow{\cdot p^{N-m}=0} \overline{\mathfrak{M}}[p^m]$ is the zero map. Consequently if we consider the transition map from the N-th level to the m-th level, we get a factorization $\mathfrak{N}_N \to \mathfrak{M}_{\mathrm{tors}}/p^m \subset \mathfrak{N}_m$, whose restriction to $\mathfrak{M}_{\mathrm{tors}}/p^N \subset \mathfrak{N}_N$ is the isomorphism mentioned above. Therefore by postcomposing with the inverse of the above isomorphism, we get a splitting $\mathfrak{N}_N \simeq \mathfrak{M}_{\mathrm{tors}} \oplus \overline{\mathfrak{M}}$. The two sequences in the beginning combine into

$$0 \to \mathfrak{N}_n \to \mathfrak{M}/p^n \to (\mathfrak{M})^{\vee\vee}/p^n \to \overline{\mathfrak{M}}/p^n \to 0.$$

This finishes the proof. □

Corollary 4.8. *Let J' be the annihilator of $\overline{\mathfrak{M}^i}$ with $i>0$, and let ρ' be such that $J'+(u)=(u,p^{\rho'})$. Then we have $\rho' \leq d(e,i-1)$.*

Proof. Since we have a natural injection $\mathfrak{M}^i/p^n \hookrightarrow \mathfrak{M}^i_n$, this follows from Lemma 4.7 and Theorem 4.4. □

Lastly we present our ultimate application:

Theorem 4.9. *There exists a constant $c(e,i)$ depending only on ramification index e and cohomological degree $i>0$, such that if the $\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C,\mathbb{Z}_p)_{\mathrm{tors}}$ is annihilated by p^m, then the $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)_{\mathrm{tors}}$ is annihilated by p^{m+c}.*

In the proof below, one finds the explicit expression $c(e,i)=2\cdot d(e,i-1)+d(e,i)$, where $d(e,j)$ is defined in Notation 2.15.

Proof. By [BS22, Theorem 1.8.(1)&(5)], we have a natural exact sequence:

$$0 \to \mathfrak{M}^i/u \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)\otimes_{W,\varphi^{-1}} W \to \mathfrak{M}^{i+1}[u] \to 0.$$

By Theorem 4.4, we see that the third term is annihilated by $p^{d(e,i)}$. We claim that $\left(\mathfrak{M}^i/u\right)_{\mathrm{tors}}$ is annihilated by $p^{m+2\cdot d(e,i-1)}$. Our theorem follows from this claim, by taking $c=2\cdot d(e,i-1)+d(e,i)$.

To see our claim: By Remark 3.1, there are two exact sequences

$$0=\mathfrak{M}^i_{\mathrm{tf}}[u] \to \mathfrak{M}^i_{\mathrm{tors}}/u \to \mathfrak{M}^i/u \to \mathfrak{M}^i_{\mathrm{tf}}/u \to 0,$$

$$0=(\mathfrak{M}^i)^{\vee\vee}[u] \to \overline{\mathfrak{M}^i}[u] \to \mathfrak{M}^i_{\mathrm{tf}}/u \to (\mathfrak{M}^i)^{\vee\vee}/u \to \overline{\mathfrak{M}^i}/u \to 0.$$

Since $(\mathfrak{M}^i)^{\vee\vee}$ is finite free, its reduction mod u is finite free over W. Therefore the second sequence above implies that $\left(\mathfrak{M}^i_{\mathrm{tf}}/u\right)_{\mathrm{tors}} \cong \overline{\mathfrak{M}^i}[u]$. Note that the term $\mathfrak{M}^i_{\mathrm{tors}}/u$ in the first exact sequence above is torsion, and hence taking the torsion submodules in this exact sequence still gives back an exact sequence. Combining the above analysis, we get a natural exact sequence:

$$0 \to \mathfrak{M}^i_{\mathrm{tors}}/u \to \left(\mathfrak{M}^i/u\right)_{\mathrm{tors}} \to \left(\mathfrak{M}^i_{\mathrm{tf}}/u\right)_{\mathrm{tors}} \cong \overline{\mathfrak{M}^i}[u] \to 0.$$

By Corollary 4.8, we see that the third term above is annihilated by $p^{d(e,i-1)}$. We have reduced our claim to: The module $\mathfrak{M}^i_{\mathrm{tors}}/u$ is annihilated by $p^{m+d(e,i-1)}$.

We have an exact sequence:

$$0 \to \mathfrak{M}^i[u^\infty] \to \mathfrak{M}^i_{\mathrm{tors}} \to \mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}} \to 0,$$

hence the following exact sequence:

$$0 \to \mathfrak{M}^i[u^\infty]/u \to \mathfrak{M}^i_{\mathrm{tors}}/u \to \mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}/u \to 0.$$

By Theorem 4.4 (with $n=\infty$), we see that the first term above is annihilated by $p^{d(e,i-1)}$. Lastly by combining [BS22, Theorem 1.8.(5) and Section 17] and [BMS18, Theorem 1.8.(iv)], we see that there is a (non-canonical) isomorphism $\mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}[1/u] \simeq \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C,\mathbb{Z}_p)_{\mathrm{tors}} \otimes_{\mathbb{Z}_p} \mathfrak{S}[1/u]$, therefore $\mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}$ is annihilated by p^m, finishing our proof. □

In the above, one could improve the constant c slightly by replacing the bound obtained in Theorem 4.4 with Theorem 4.2 at several places. However we feel that the constant c obtained via this method is unlikely to be optimal anyway, so we do not choose to optimize the bound in the proof to prevent complicating notations.

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